Interactions between noncommutative algebra and algebraic geometry

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Overview of the Field

The root of noncommutative algebra goes back to 1843 when Hamilton discovered the quaternions. The subject of abstract algebra has been developed since the early twentieth century by Wedderburn, Artin, Brauer, Noether, and later by Amitsur, Jacobson, Kaplansky, Goldie, Herstein and many others. New research topics in noncommutative algebra and its interplay with other fields such as Lie theory, geometry and physics has emerged in recent years. The structure of noncommutative algebras has been understood by the use of algebraic, combinatorial, geometric and homological means.

Noncommutative algebraic geometry is an interdisciplinary subject that arises from the interaction between noncommutative algebra and algebraic geometry. Originated by Artin, Schelter, Tate and Van den Bergh [6, 9, 10], noncommutative projective geometry has grown from the basic principle that the global techniques of classical projective algebraic geometry furnish powerful methods and intuition for the study of noncommutative graded algebras. Specifically, the category of graded modules modulo those of finite length over a noetherian connected graded algebra form an appropriate analogue for the category of coherent sheaves on a projective variety [12]. This technique has been particularly successful in understanding important classes of noncommutative algebras.

Over the last fifteen years, noncommutative projective curves have been "understood" by Artin and Stafford [7, 8] in terms of graded algebras of Gelfand-Kirillov dimension two and by Reiten and Van den Bergh [40] in terms of proper categories of global dimension two. More recently, noncommutative projective surfaces have been studied extensively by many researchers. The birational theory of noncommutative projective surfaces was extended in the work of Chan and his coauthors [23, 24, 25, 27, 26] using ideas of Mori's minimal model program. The moduli of Azumaya algebras on surfaces have been studied by Artin and de Jong [5]. Various explicit and interesting examples of noncommutative surfaces have been examined. Regular algebras of global dimension three were classified by Artin, Schelter, Tate and Van den Bergh in 1980's [6, 9, 10].

There have been many recent developments in the classification of regular algebras of global dimension four (or their associated quantum projective three spaces) in papers by Lu, Palmieri, Shelton, Stephenson Vancliff, Wu and Zhang [44, 45, 35, 53] and others.

Many new techniques have been introduced and used in noncommutative algebra and noncommutative algebraic geometry. Also applications of noncommutative algebraic geometry have been found. Stacks, derived categories, tilting and dualizing complexes are currently standard tools in the subject. The purpose of this workshop was to discuss various aspects of the interaction between noncommutative algebra and algebraic geometry, including the latest developments in noncommutative algebraic geometry and their applications.

Recent Developments and Open Problems

Recent Developments

An active research topic in the subject, with many participants, remains the classification of noncommutative surfaces (both in the setting of orders and of highly noncommutative algebras) and higher dimensional noncommutative analogues of projective space. This has again lead to interesting new concepts like naïve blow-ups [33], double Ore extensions [59] and their generalizations. It has also lead to various generalizations of Koszul duality for non-quadratic algebras [17]. Some of the best-understood noncommutative surfaces are the del Pezzo surfaces introduced by Van den Bergh in his work on noncommutative blow-ups [55]. They have resurfaced in recent work of Auroux, Katzarkov and Orlov on homological mirror symmetry [13]. New constructions of such surfaces have recently appeared in work of Etingof and Ginzburg [28] and these are, in turn, related to graded Calabi-Yau algebras.

Developments in noncommutative algebraic geometry provide new ideas and even new research directions to commutative algebra, noncommutative algebra and algebraic geometry. Many basic concepts, including rigid dualizing complexes, strongly noetherian rings, twisted homogeneous coordinate rings, point varieties and $\mathbf{Z}$-algebras, that first appeared in noncommutative algebraic geometry, are now routinely used in other contexts. For example, rigid dualizing complexes have been used to give a completely new approach to Grothendieck duality in the commutative case [58], as well as in the study of Noetherian noncommutative Hopf algebras [20]. In another direction, $\mathbf{Z}$-algebra techniques are fundamental to the work on graded twisted Weyl algebras [47] and Cherednik algebras [31, 32]. As an illustration of the applications to the other areas, we mention three different papers. The first is the work of Van den Bergh on noncommutative crepant resolutions [56] that establishes a noncommutative model for resolution of singularities and avoids the technical issue of blowing up in commutative algebraic geometry. Noncommutative crepant resolutions have been used in both commutative algebra and algebraic geometry. The second is the paper by Ben-Zvi and Nevins [16] which used the techniques of noncommutative algebraic geometry to provide a bridge between KP soliton equations and Calogero-Moser many-body systems. The ramifications of this work and related moduli questions remains an active area of research. A recent work of Okounkov on random surfaces [39] gave an surprising application of noncommutative algebraic geometry to the study of difference equations and probability, which could further relate this new subject to a broader aspect of mathematics.

Noncommutativity has proved to be stacky. Recently, algebraic stacks were used by Lieblich to study finite dimensional division algebras over fields of transcendence degree two [34], by Chan to study noncommutative projective schemes, and by Reichstein to study essential dimension [21]. The theory of algebraic stacks (and possible generalizations) provides a powerful idea/method for a large class of noncommutative rings and schemes with big centers, including noncommutative Calabi-Yau algebras. Examples show that the stacky structure can be realized by using appropriate categories, which agrees with the basic principle of noncommutative algebraic geometry. Further research on algebraic stacks would be beneficial to the subject.

Ideas from representation theory play an important role in the recent developments. Derived Categories are used more and more, and there are many examples of commutative and noncommutative varieties who share their derived category with representations of a noncommutative finite dimensional algebra (in a slightly more general setting, the finite dimensional algebras should be replaced by finite dimensional DG-algebras or finite dimensional A_∞-algebras). Invariants defined by using derived categories such as Rouquier's dimension, thick subcategories, t-structures, and rigid dualizing complexes would have significant impacts to the study of noncommutative algebra and geometry.

Another important advance has been in the study of combinatorial aspects of noncommutative algebra. Bell recently proved that an affine prime Goldie algebra of quadratic growth is either primitive or satisfies a polynomial identity [14], answering a famous question of Small in the affirmative. Smoktunowicz has proved a "gap" theorem stating that there is no connected graded affine domain with Gelfand-Kirillov dimension between 2 and 3 in [52]. Centralizers of elements in algebras of low Gelfand-Kirillov dimension have been studied in detail [15, 51].

Open Questions

Here is a partial list of open questions in the field.

1. Artin's conjecture and classification of noncommutative projective surfaces.

In 1992 Artin conjectured that every division algebra of transcendence degree two that comes from a noncommutative projective surface is either finite over its center, q-ruled, or q-rational (including the Sklyanin division ring) which correspond to all known noncommutative projective surfaces up to birational equivalence [4]. Artin proved that these division algebras are non-isomorphic by showing that their prime divisors are different [4]. This conjecture is very natural after the landmark work of Artin and Stafford [7, 8] that classified all noncommutative projective curves (or noncommutative graded domains of Gelfand-Kirillov dimension two). As mentioned above a great deal of research has been motivated by Artin's conjecture: the work of Chan, Hacking, Ingalls, Kulkarni, and Nyman [24, 25, 23, 27, 26], the manuscript by Artin and de Jong [5], and the series of papers by Keeler, Rogalski, Stafford and Sierra [33, 41, 42, 43, 46], and Van den Bergh's theory of monoidal transformations [55]. Artin's conjecture is the first step towards the classification of noncommutative projective surfaces. At this point Artin's conjecture is still one of the main open questions in noncommutative algebraic geometry and the classification of noncommutative projective surfaces seems a long way off.

2. Classification of quantum $\mathbf{P}^n$.

This is another major open question in the field. A subquestion is the classification of (Artin-Schelter) regular algebras of global dimension large than three. Since the classification of regular algebras of global dimension three in 1980s, many researchers have been interested in regular algebras of global dimension four. For example, a recent paper by Cassidy-Vancliff introduced the notion of a graded skew Clifford algebra [22] aiming at constructing more regular algebras with finite point modules. Lu, Palmieri, Wu and Zhang used A_∞-algebra methods to attack this problem [35]. One special class of regular algebras is so-call Sklyanin algebras [48, 49] that have been understood by Odesskii, Feigin, Smith, Stafford, Tate and Van den Bergh [37, 38, 50, 54]. The general classification has not been finished, even for global dimension four.

3. Geometric aspect of quantum groups.

Quantum groups (or noncommutative and noncocommutative Hopf algebras) have been studied extensively for more than twenty years and their algebraic and representation-theoretic aspects are understood to a large degree. However, the geometric aspect of quantum groups is less well-understood. Using cohomology, the geometry of the Koszul dual of some special quantum groups were studied by Ginzburg and Kumar in [30]and by Friedlander and Suslin in [29]. A more direct approach to the question has not been established. Ideas and techniques in noncommutative algebraic geometry should be useful in this study. Another interesting open question about quantum groups and Hopf algebras is Brown's conjecture: Every noetherian affine Hopf algebra has finite injective dimension [18, 19]. A partial answer is given in the PI case [57].

4. Ideals theory of Iwasawa algebras.

Noncommutative Iwasawa algebras have repeatedly been used in Iwasawa theory, which is probably the best general technique in arithmetic geometry for studying the mysterious relations between exact arithmetic formulae and special values of L-functions. The Iwasawa algebras Γ_G and their companion Ω_G form an interesting class of noncommutative complete semilocal noetherian algebras. For simplicity we assume that the compact p-adic analytic group G is uniform. In this case both Γ_G and Ω_G are local and have various nice homological properties – see the survey paper by Ardakov and Brown [1]. The structure of Iwasawa algebras and modules over them have been studied extensively by several authors from an algebraic point of view and one of the main questions in this research direction is the Ardakov-Brown Question [1]: Let G be almost simple and uniform. Are there prime ideals of Ω_G other than zero and the maximal ideal? This was verified only for G being congruence subgroups of $SL_2(\mathbf{Z}_p)$ [2, 3]. New techniques are needed for handling the higher dimensional cases.

5. Dimension theory in noncommutative algebra and noncommutative algebraic geometry.

One difficult issue in noncommutative algebra/geometry is to find an effective dimension theory. Some generalizations of commutative dimension theory such as Krull dimension do not give the correct numbers in the noncommutative case, while others are not easy to compute. Gelfand-Kirillov dimension is a commonly used dimension function and is relatively easy to compute in many cases. One open question about Gelfand-Kirillov dimension is: under what reasonable hypotheses, is it an integer? The recent works of Bell [14] and Chan [27] are dependent on a good behavior of a dimension function.

Presentation Highlights

Speaker: **Jason Bell** (Simon-Fraser University)
Title: *Centralizers in finitely generated algebras and in division algebras*
Abstract: We look at the problem of describing centralizers in algebras and in division algebras. Our main result is that if A is a finitely generated complex noetherian algebra of GKdimension strictly less than 4 then the centralizer of a non-scalar element satisfies a polynomial identity. We also look at the connection between transcendence degree of subfields and the size of centralizers in division algebras and formulate a few conjectures.

Speaker: **Daniel Chan** (University of New South Wales)
Title: *A non-commutative Mori contraction*
Abstract: One of M. Artin's conjectures can be loosely stated as: a noncommutative surface is, up to birational equivalence, either ruled or finite over its centre. This suggests that it would be nice to have a criterion for a noncommutative surface to be ruled. In the commutative case, there is a criterion based on Mori theory. Given a K-negative extremal ray C in the cone of curves with self-intersection zero, the surface is ruled. In this talk, we discuss the possibility of a noncommutative version of this result. We show that given a noncommutative smooth projective surface (appropriately defined) which contains an object like the ray C above, there is a non-commutative "Mori contraction" to a curve. This is a report on joint work with Adam Nyman.

Speaker: **Michele D'Adderio** (University of California at San Diego)
Title: *On isoperimetric profiles of algebras*
Abstract: Isoperimetric profile in algebras was first introduced by Gromov. In this talk we show the behavior of the isoperimetric profile under various ring theoretic constructions and its relation with amenability. We show that the isoperimetric profile is a finer invariant than the lower transcendence degree, and we use it to answer a question of J.J. Zhang.

Speaker: **Birge Huisgen-Zimmermann** (University of California at Santa Barbara)
Title: *Generic representations of quivers with relations*
Abstract: The irreducible components of varieties parametrizing the finite dimensional representations of a finite dimensional algebra A are explored, in terms of both their geometry and the structure of the modules they encode; expected close connections between the two aspects are rendered more explicit. In particular, we establish the existence and uniqueness (not up to isomorphism, but in a strong sense to be specified) of generic modules, that is, of modules which display all categorically defined generic properties of the modules parametrized by a given irreducible component. Our approach to existence is largely constructive, by way of minimal projective presentations. We follow with an investigation of the properties of such generic modules with regard to quiver and relations of A. The sharpest specific results on all fronts are obtained for truncated path algebras, that is, path algebras of quivers modulo ideals generated by all paths of a fixed length.

Speaker: **Osamu Iyama** (Nagoya University)
Title: *Cluster tilting in* 2*-Calabi-Yau categories*
Abstract: Cluster tilting theory reveals combinatorial structure of 2-Calabi-Yau triangulated categories, and is applied to categorify Fomin-Zelevinsky cluster algebras by many authors (Buan, Marsh, Reineke, Reiten Todorov, Caldero, Chapoton, Schiffler, Keller,...).In my talk, we will introduce cluster tilting theory in 2-Calabi-Yau triangulated category. In particular, a combinatorial description of the change of endomorphism algebras of cluster tilting objects via mutation process is given in terms of Fomin-Zelevinsky quiver mutation rule and Derksen-Weyman-Zelevinsky quiver with potential mutation rule.

Speaker: **Ulrich Kraehmer** (University of Glasgow)

Title: *On the Hochschild (co)homology of quantum homogeneous spaces*

Abstract: In this talk I will speak about a generalisation of a result by Brown and Zhang establishing Poincare duality in Hochschild (co)homology for a class of algebras that can be considered as noncommutative analogues of affine homogeneous spaces.

Speaker: **Max Lieblich** (Princeton University)

Title: *The period-index problem for surfaces over finite fields*

Abstract: I will discuss the period-index problem for Brauer groups of fields of transcendence degree 2 over finite fields, and how stacky techniques relate this problem to the geometry of moduli spaces of vector bundles and to the Hasse principle for geometrically rational varieties over global fields.

Speaker: **Valery Lunts** (Indiana University)

Title: *Categorical resolution of singularities*

Abstract: This is my work in progress. I want to propose the notion of a categorical resolution of singularities, compare it with the usual resolutions in algebraic geometry and discuss some examples and conjectures.

Speaker: **Susan Montgomery** (University of Southern California)

Title: *Recent progress in "Invariant Theory" for Hopf algebras*

Abstract: Let H be a finite dimensional Hopf algebra acting on an algebra A over a field k. Recently a number of questions about the relationship between A, its subalgebra of invariants A^H, and the semi-direct product $A\#H$, open since the 1970's and 1980's, have been solved. For example: (1) if H is semisimple, then the Jacobson radical is always H-stable (this is work of Linchenko); (2) if H is semisimple and A is H-semiprime, then any H-stable left or right ideal of A intersects A^H non-trivially. I will survey some of these results and discuss their proofs.

Speaker: **Zinovy Reichstein** (University of British Columbia)

Title: *Essential dimension and algebraic stacks*

Abstract: The essential dimension of an algebraic object (e.g., of an algebra, a quadratic form, or an algebraic variety) is the minimal number of independent parameters required to define the underlying structure. This numerical invariant has been studied by a variety of algebraic, geometric and cohomological techniques. In this talk, based on joint work with P. Brosnan and A. Vistoli, I will discuss a new approach based on the notion of essential dimension for an algebraic stack.

Speaker: **Daniel Rogalski** (University of California at San Diego)

Title: *Subalgebras of the Sklyanin Algebra*

Abstract: We study subalgebras of the 3-dimensional Sklyanin algebra S, particularly those generated in degree 3. We classify such algebras which are also maximal orders. Geometrically, each such algebra 4 behaves like a blowup of the Sklaynin projective plane along a divisor of degree at most 7 on the elliptic curve.

Speaker: **David Saltman** (University of Texas at Austin)

Title: *Quaternion algebras and their maximal subfields*

Abstract: Using the generic splitting field of Amitsur, we know that two division algebras with the same splitting fields are quite close: powers of each other in the Brauer group. However, if you restrict to finite dimensional splitting fields, or even maximal subfields, the situation is much different. Over global fields, two division algebras of degree greater than 2 can have the same finite dimensional splitting fields and not be powers of each other. However, two quaternion algebras over a global field with the same maximal subfields are isomorphic. Several people asked whether this is true in general. We will show that if youassume the center field F has zero unramified Brauer group, and D/F, D'/F are quaternion algebras with the same maximal subfields, then $D \cong D'$. We will also discuss a generalization to higher cohomology due to Skip Garibaldi.

Speaker: **Brad Shelton** (University of Oregon)

Title: *Noncommutative Koszul Algebras from Combinatorics and Topology*

Abstract: We consider a construction given by Gelfand, Retakh, Serconek and Wilson, that builds noncommutative graded quadratic algebras from finite layered graphs. A mistake in the literature suggests that all such algebras are Koszul. We give some results on the Koszul property of these algebras when the associated graph is related to a regular CW-complex.

Speaker: **Susan Sierra** (University of Washington)

Title: **The classification of birationally commutative surfaces**

Abstract: We give a complete classification of birationally commutative projective surfaces (connected $\mathbf{N}$-graded noetherian domains of GKdimension 3 that are birational to a commutative surface in Artin's sense), and show that all such algebras fall into four families determined by geometric data. We relate the geometry of the underlying data and the algebraic properties of the algebras, and discuss potential generalizations to higher dimensions. This extends results of Rogalski and Stafford in the case that the algebra is generated in degree 1.

Speaker: **Agata Smoktunowicz** (University of Edinburgh)

Title: GKdimension of factor algebras of Golod-Shafarevich algebras

Abstract: It is known that Golod-Shafarevich algebras have exponential growth. In this talk it is shown that also all non-nilpotent factor rings of generic Golod-Shafarevich algebras over fields of infinite transcendence degree have exponential growth, provided that the number of defining relations of degree less then n grows exponentially with n. This answers a question stated by Efim Zelmanov in the paper "Some open problems in the theory of infinite dimensional algebras".

Speaker: **Toby Stafford** (University of Manchester)

Title: *Applications of noncommutative geometry to Cherednik algebras*

Abstract: This talk will report on joint work with Victor Ginzburg and Iain Gordon. We establish a link between two geometric approaches to the representation theory of rational Cherednik algebras of type A: one based on noncommutative geometry; the other involving quantum hamiltonian reduction of an algebra of differential operators. This link is achieved by showing that the process of hamiltonian reduction intertwines a naturally defined geometric twist functor on D-modules with the shift functor for the Cherednik algebra. If I have time I will also apply this to the structure of characteristic cycles of modules over the Cherednik algebra.

Speaker: **Hokuto Uehara** (Tokyo Metropolitan University)

Title: *Tilting generators via ample line bundles*

Abstract: We construct tilting generators by ample line bundles under some assumptions.

Speaker: **Quanshui Wu** (Fudan University)

Title: *Non-commutative Castelnuovo-Mumford Regularity and AS regular Algebras*

Abstract: Let A be a connected graded k-algebra generated in degree 1, with a balanced dualizing complex. I will talk about the Castelnuovo-Mumford regularity and the Ext regularity and prove that these regularities coincide for all finitely generated A-modules if and only if that A is a Koszul AS-regular algebra. By using Castelnuovo-Mumford regularity, we also prove that any Koszul standard AS-Gorenstein algebra is AS-regular.

Scientific Progress Made

The meeting allowed people working in noncommutative algebraic geometry and related fields to become up to date with current progress in the field. There were also many discussions between participants yielding new research projects. This meeting had many younger speakers and participants with new connections to noncommutative algebraic geometry. There has been much recent progress in the area and the meeting was an important part of such progress.

Outcome of the Meeting

This workshop brought together 38 researchers (including three Ph.D. students) from various countries in different areas of noncommutative ring theory, algebraic geometry, representation theory and the interdisciplinary subject noncommutative algebraic geometry. There were many interesting talks and engaging discussions during the workshop. The collaborations between participants were fruitful which led some joint research projects and some of which resulted joint publications. Since this workshop, there have been a few workshops in the same topics around the world:

University of Manchester, Manchester, UK from August 3-7, 2009. RIMS, Kyoto University, Kyoto, Japan from August 24-28, 2009, Oberwolfach, Germany, May 9-15, 2010 (planed). More researchers in the related subjects have become interested in noncommutative algebraic geometry. This workshop definitely created new mathematics in noncommutative algebraic geometry and will further expand the subject.

List of Participants

Bell, Jason (Simon Fraser University)
Chan, Daniel (Univesity of New South Wales)
Chan, Kenneth (University of New South Wales)
D'Adderio, Michele (University of California - San Diego)
Goodearl, Kenneth (University of California, Santa Barbara)
Green, Ed (Virginia Tech)
Huisgen-Zimmermann, Birge (University of California, Santa Barbara)
Ingalls, Colin (University of New Brunswick)
Ishii, Akira (Hiroshima University)
Iyama, Osamu (Nagoya University)
Kawamata, Yujiro (The University of Tokyo)
Kirkman, Ellen (Wake Forest University)
KrŁhmer, Ulrich (University of Glasgow)
Kulkarni, Rajesh (Michigan State University)
Letzter, Edward (Temple University)
Lieblich, Max (Princeton University)
Lorenz, Martin (Temple University)
Lu, Di-Ming (Zhejiang University)
Lunts, Valery (Indiana University)
Montgomery, Susan (University of Southern California)
Mori, Izuru (Shizuoka University)
Nyman, Adam (University of Montana)
Reichstein, Zinovy (University of British Columbia)
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Shelton, Brad (University of Oregon)
Sierra, Susan (University of Washington)
Small, Lance (University of California, San Diego)
Smoktunowicz, Agata (University of Edinburgh)
Stafford, Toby (University of Michigan)
Todorov, Gordana (Northeastern University)
Uehara, Hokuto (Tokyo Metropolitan University)
Vancliff, Michaela (University of Texas at Arlington)
Vonessen, Nikolaus (University of Montana)
Wu, Quanshui (Fudan University)
Yekutieli, Amnon (Ben Gurion University)
Zhang, James (University of Washington)
Zhang, Jun (University of Washington)

References

[1] K. Ardakov and K.A. Brown, Ring-theoretic properties of Iwasawa algebras: a survey. Doc. Math. (2006), Extra Vol., 7–33.

[2] K. Ardakov, F. Wei and J.J. Zhang, Reflexive ideals in Iwasawa algebras, *Adv. in Math.*, **218** (2008), no. 3, 865–901.

[3] K. Ardakov, F. Wei and J.J. Zhang, Nonexistence of reflexive ideals in Iwasawa algebras of Chevalley type, *J. of Algebra*, **320** (2008), 259-275.

[4] M. Artin, Geometry of quantum planes, *Azumaya algebras, actions, and modules* (Bloomington, IN, 1990), 1–15, Contemp. Math., 124, AMS, Providence, RI, 1992.

[5] M. Artin and A.J. de Jong, "Stable Orders over Surfaces", in preparation.

[6] M. Artin and W. Schelter, Graded algebras of global dimension 3, *Adv. in Math.* **66** (1987), no. 2, 171–216.

[7] M. Artin and J. T. Stafford, Noncommutative graded domains with quadratic growth, *Invent. Math.,* **122** (1995) 231-276.

[8] M. Artin and J. T. Stafford, Semiprime graded algebras of dimension two, *J. Algebra* **227** (2000), no. 1, 68–123.

[9] M. Artin, J. Tate, and M. Van den Bergh, Some algebras associated to automorphisms of elliptic curves, *The*

Grothendieck Festschrift, Vol. I, 33–85, Progr. Math., 86, Birkhäuser Boston, Boston, MA, 1990.

[10] M. Artin, J. Tate, and M. Van den Bergh, Modules over regular algebras of dimension 3, *Invent. Math.* **106** (1991), no. 2, 335–388.

[11] M. Artin, and M. Van den Bergh, Twisted homogeneous coordinate rings, *J. Algebra* **133** (1990), no. 2, 249–271.

[12] M. Artin and J.J. Zhang, Noncommutative projective schemes, *Adv. Math.* **109** (1994), no. 2, 228–287.

[13] D. Auroux, L. Katzarkov, and D. Orlov, Mirror symmetry for del Pezzo surfaces: vanishing cycles and coherent sheaves, *Invent. Math.* **166** (2006), 537–582.

[14] J.P. Bell, A dichotomy result for prime algebras of Gelfand-Kirillov dimension two, preprint.

[15] J.P. Bell, Centralizers in Domains of Finite Gelfand-Kirillov Dimension Bull. Lond. Math. Soc. **41** (2009), no. 3, 559–562.

[16] D. Ben-Zvi and T. Nevins, From solitons to many-body systems, *Pure Appl. Math. Q.* **4** (2008), 319–361.

[17] R. Berger, Koszulity for nonquadratic algebras, *J. Algebra* **239** (2001), 705–734.

[18] K.A. Brown, Representation theory of Noetherian Hopf algebras satisfying a polynomial identity, *Trends in the representation theory of finite-dimensional algebras* (Seattle, WA, 1997), 49–79, Contemp. Math., 229, AMS, Providence, RI, 1998.

[19] K.A. Brown, Noetherian Hopf algebras. Turkish J. Math. **31** (2007), suppl., 7–23.

[20] K.A. Brown and J.J. Zhang, Dualising complexes and twisted Hochschild (co)homology for Noetherian Hopf algebras, J. Algebra **320** (2008), no. 5, 1814–1850.

[21] P. Brosnan, Z. Reichstein and A. Vistoli, Essential dimension, spinor groups and quadratic forms Annals of Mathematics,to appear.

[22] T. Cassidy and M. Vancliff Generalizations of graded Clifford algebras and of complete intersections, J. London Math. Soc. (2) **81** (2010) 91-112.

[23] D. Chan, P. Hacking and C. Ingalls, Canonical singularities of orders over surfaces, Proc. Lond. Math. Soc. (3) **98** (2009), no. 1, 83–115.

[24] D. Chan and C. Ingalls, Noncommutative coordinate rings and stacks, *Proc. London Math. Soc.* (3) **88** (2004), no. 1, 63–88.

[25] D. Chan and C. Ingalls, The minimal model program for orders over surfaces, Invent. Math. **161** (2005), no. 2, 427–452.

[26] D. Chan and R.S. Kulkarni, Numerically Calabi-Yau orders on surfaces, J. London Math. Soc. (2) **72** (2005), no. 3, 571–584.

[27] D. Chan and A. Nyman, Non-commutative Mori contractions and $\mathbf{P}^1$-bundles, preprint, 2009.

[28] P. Etingof and V. Ginzburg, Noncommutative del Pezzo surfaces and Calabi-Yau algebras, preprint (2007) arXiv:0709.3593.

[29] E.M. Friedlander and A. Suslin, Cohomology of finite group schemes over a field, Invent. Math. **127** (1997), no. 2, 209-270.

[30] V. Ginzburg and S. Kumar, Cohomology of quantum groups at roots of unity, Duke Math. J. **69** (1993), no. 1, 179-198.

[31] I. Gordon and J.T. Stafford, Rational Cherednik algebras and Hilbert schemes. *Adv. Math.* **198** (2005), 222-274.

[32] I. Gordon and J.T. Stafford, Rational Cherednik algebras and Hilbert schemes, II: Representations and sheaves, *Duke Math. J.* **132** (2006), 73–135.

[33] D.S. Keeler, D. Rogalski and J.T. Stafford, Nave noncommutative blowing up. Duke Math. J. 126 (2005), no. 3, 491–546.

[34] M. Lieblich, The period-index problem for fields of transcendence degree 2 preprint.

[35] D.-M. Lu, J. Palmieri, Q.-S. Wu and J.J. Zhang, Regular algebras of dimension 4 and their A_∞-Ext-algebras. *Duke Math. J.* 137 (2007), no. 3, 537–584.

[36] D.-M. Lu, J. Palmieri, Q.-S. Wu and J.J. Zhang, A-infinity structure on the Ext-algebras, *Journal of pure and applied algebra*, **213** (2009), no. 11, 2017–2037.

[37] A.V. Odesskii and B.L. Feigin, Sklyanin algebras associated with an elliptic curve (in Russian), Preprint, Institute for Theoretical Physics, Kiev, (1989).

[38] A.V. Odesskii and B.L. Feigin, Elliptic Sklyanin algebras (in Russian), Funk. Anal. Prilozh., 23(3) (1989), 45-54.

[39] A. Okounkov, Noncommutative geometry of random surfaces preprint, 2009.

[40] I. Reiten and M. Van den Bergh, Noetherian hereditary abelian categories satisfying Serre duality. J. Amer. Math. Soc. **15** (2002), no. 2, 295–366.

[41] D. Rogalski, Generic noncommutative surfaces. Adv. Math. 184 (2004), no. 2, 289–341

[42] D. Rogalski and J.T. Stafford, Nave noncommutative blowups at zero-dimensional schemes. J. Algebra 318 (2007), no. 2, 794–833

[43] D. Rogalski and J. T. Stafford, A class of noncommutative projective surfaces, *Proc. London Math. Soc.*, (3) **99** (2009), no. 1, 100–144.

[44] B. Shelton and M. Vancliff, Some Quantum $\mathbf{P}^3$s with One Point, *Comm. Alg.* **27** No. 3 (1999), 1429-1443.

[45] B. Shelton and M. Vancliff, Embedding a Quantum Rank Three Quadric in a Quantum $\mathbf{P}^3$, *Comm. Alg.* **27** No. 6 (1999), 2877-2904.

[46] S.J. Sierra, *The geometry of birationally commutative graded domains,* thesis, University of Michigan, 2008.

[47] S.J. Sierra, G-algebras, twistings, and equivalences of graded categories, Algebr. Represent. Theory, in press, arXiv: math/0608791, 2006.

[48] E. K. Sklyanin, Some algebraic structures connected with the Yang-Baxter equation (Russian), *Funktsional. Anal. i Prilozhen* **16** (1982), no. 4, 27–34.

[49] E. K. Sklyanin, Some algebraic structures connected with the Yang-Baxter equation. Representations of a quantum algebra, (Russian) *Funktsional. Anal. i Prilozhen* **17** (1983), no. 4, 34–48.

[50] S. P. Smith and J. T. Stafford, Regularity of the four-dimensional Sklyanin algebra, *Compositio Math.* **83** (1992), no. 3, 259–289.

[51] A. Smoktunowicz, Centers in domains with quadratic growth. Cent. Eur. J. Math. **3** (2005), no. 4, 644-653.

[52] A. Smoktunowicz, There are no graded domains with GK dimension strictly between 2 and 3, *Invent. Math.* **164** (2006), 635–640.

[53] D.R. Stephenson and M. Vancliff, Finite quantum P3s that are infinite modules over their centers, *J. Algebra*, in press.

[54] J. Tate and M. Van den Bergh, Homological properties of Sklyanin algebras, Invent. Math. **124** (1996), no. 1-3, 619–647.

[55] M. Van den Bergh, Blowing up of non-commutative smooth projective surfaces, *Mem. Amer. Math. Soc.,* **154** (2001), no. 734.

[56] M. Van den Bergh, Non-commutative crepant resolutions, *The legacy of Niels Henrik Abel*, 749–770, Springer, Berlin, 2004.

[57] Q. Wu and J.J. Zhang, Noetherian PI Hopf algebras are Gorenstein, *Trans. Amer. Math. Soc.* **355** (2003), 1043-1066.

[58] A. Yekutieli and J.J. Zhang, Rigid complexes via DG algebras, *Trans. Amer. Math. Soc.* **360** (2008), 3211–3248.

[59] J.J. Zhang and J. Zhang, Double Extension Regular Algebras of Type (14641), *J. Algebra*, **322** (2009), 373–409.