Locally symmetric spaces

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1 The setting

A locally symmetric space is the quotient $\Gamma\backslash X$ of a globally symmetric space $X = G/K$, where G is a non compact reductive Lie Group with maximal compact subgroup K and where Γ is a discrete subgroup of G. Locally symmetric spaces are important in geometry, analysis, and as well as in number theory. We considered at the conference both arithmetic and non-arithmetic subgroups Γ as well as compact and non-compact locally symmetric spaces $\Gamma\backslash X$.

The topics covered at the conference belong to

- compactifications,
- differential geometry and topology, and
- arithmetic.

Techniques and results from all these areas were represented.

2 Background material

Several developments in different fields over the past few years made this an opportune time for this workshop.

2.1 Compactifications

When X/Γ is non-compact, various compactifications (often singular) have been introduced to address different problems in geometry and number theory. The construction and relationship between various compactifications has become more clear through the work of several mathematicians, particularly Borel, Goresky, Ji, and Zucker.

Various types of cohomology associated to these compactifications are important in number theory. The relation between the L^2-cohomology and the intersection cohomology of the Baily-Borel compactification is important for Langlands's program and was resolved by Looijenga and Saper-Stern. The further relationship between this and the weighted cohomology of the (less singular) reductive Borel-Serre compactification was settled by Goresky, Harder, and MacPherson. The analogous theorem for the intersection cohomology of the reductive Borel-Serre compactification was a conjecture of Rapoport which was established by Saper.

More recently the relation between the space of L^2-harmonic forms and the intersection cohomology of the reductive Borel-Serre compactification was determined by Saper in a general context. The presence of the reductive Borel-Serre compactification in recent work shows its ubiquity.

A topological problem concerning locally symmetric spaces which has received a fair bit of attention is the problem to determine if a cycle class of a locally symmetric subvariety is a nontrivial cohomology class; see for example the work of Bergeron, Clozel, Rohlfs-Speh, Speh-Venkataramana, and Venkataramana. This problem is related on the one hand to the difficult problem of determining the restriction to a semisimple subgroup H of an irreducible representations of G as in the work of Kobayashi-Oda and on the other hand to the arithmetic problem of period integrals with respect to H of automorphic representations where special values of L-functions also play a role. Another interesting and important problem, where similar techniques are useful, are non-vanishing results for cup products of cohomology classes such as those of Bergeron and Venkataramana.

2.2 Differential geometry and topology

Important geometric invariants of locally symmetric spaces which have been considered are the analytic torsion and the related length spectrum of closed geodesics, as well as the special values of the geometric theta functions; in particular we note the work of Deitmar, Juhl, and Rohlfs-Speh. In this context, invariants of non-arithmetic subgroups are also of great interest as in the work of Bunke, Olbricht, and Leuzinger.

2.3 Arithmetic

The application of locally symmetric spaces to arithmetic involves a great deal of analysis. One of the main problems in analysis on locally symmetric spaces is the study of the spectrum of the Laplace operator on the space X/Γ. A very well-known spectral problem is to obtain a lower bound on the spectrum of the Laplacian. For GL_n, the Ramanujan conjecture is equivalent to such a bound. For $n = 2$, the best bound is due to Shahidi and Kim. In the general case, substantial work has been done by P. Sarnak and his collaborators. The techniques used here are L-functions and the lifting of automorphic representations.

Another source of interesting analytic problems is connected to the Arthur-Selberg trace formula. Particularly notable here is the recent work of W. Müller on the spectral side of the Arthur-Selberg trace formula and his results about Weyl's law for the cuspidal spectrum.

Other connections to number theory also play an important role in the study of locally symmetric spaces. This can be seen for example in the work of Ash, Bruinier, Clozel, Emerton, Hanamura, Harder, Kudla, Mahnkopf, Rapoport, Rohlfs, Schwermer and Speh. Conjectures of Langlands, Tate, Beilinson and others are the driving force behind these developments which involve Hecke eigen-functions, L-functions, special values of L-functions, special points of varieties, modular symbols, mixed motives, Chow groups, and K-theory.

3 The conference

The conference was attended by 31 scientists from all over the world. There were 17 talks, each of one hour, covering recent developments in the field. One goal of the workshop was to give an opportunity for young researchers to learn more about the different aspects of the field, the different methodologies, and the many open problems. With around half the participants being young researchers, this goal was well achieved. Another goal was to stimulate new developments, possibly involving interactions of researchers from different areas. The format of the conference, with sufficient unstructured time for informal discussions, allowed this goal to be achieved as well. Mathematical discussions, both formal and informal, in the common room continued far into the night.

3.1 Abstracts of talks (in alphabetical order)

Speaker: *Yves Benoist* (University of Paris, South)
Title: Effective equidistribution of S-integral points on symmetric varieties

Abstract: Let S be a finite set of places of a global field K. We describe counting and equidistribution results for the S-integral points on a symmetric variety defined over K. We give also an upper bound for the error term in characteristic 0. This joint work with Hee Oh is based on a polar decomposition of p-adic symmetric spaces.

Speaker: *Mathieu Cossutta* (University of Paris, 7)
Title: Asymptotics of L^2-Betti numbers in congruence coverings of some arithmetically defined locally symmetric varieties.
Abstract: Let $G/\mathbb{Q}$ be an algebraic group and $X = G(\mathbb{R})/K$ the associated symmetric space. The aim of my talk is to give some new informations on $b_{i,2}(\Gamma(p^n)\backslash X)$ when n goes to infinity for a large family of groups G, in the direction of conjectures formulated by Xue and Sarnak. Our proof is based on theta correspondence. For example when G = Sp_{2g}, the primitive holomorphic cohomology appears only in degree $r(2gr)$ for r an integer between 1, . . . , g. We then obtain that for $g \leq \frac{5}{4}p + 1$:

$$\begin{aligned}
&\mathrm{Vol}(\Gamma(p^n)\backslash X)^{\frac{2r}{g+1}(1-\frac{r(r+1)}{g(g+1)})-\epsilon} \\
&\qquad \ll_\epsilon \dim H^{r(2g-r),0}_{\text{prim},2}(\Gamma(p^n)\backslash X, \mathbb{C}) \\
&\qquad \ll_\epsilon \mathrm{Vol}(\Gamma(p^n)\backslash X)^{\frac{2r}{g+1}(1+\frac{r(r-1)}{g(g+1)})+\epsilon}
\end{aligned}$$

Speaker: *Anton Deitmar* (University of Tübingen)
Title: Automorphic forms of higher order
Abstract: The present talk presents an attempt to study them in the general context of higher order invariants and cohomology. It is shown that for arithmetic groups, higher order cohomology can be computed as $(\mathbf{g}, K)$-cohomology, even using functions of moderate growth. It is, however, an open question whether the higher order analogue of Franke's theorem holds, which states that the cohomology can be computed using automorphic forms. An action of the Hecke algebra is introduced in which the Hecke operators are bounded operators. Questions about their spectral decomposition are raised.

Speaker: *Jens Funke* (New Mexico State University)
Title: Special cohomology classes arising from the Weil representation
Abstract: The Weil representation is a well-known tool to study arithmetic and cohomological aspects of orthogonal groups. We construct certain special cohomology classes for orthogonal groups O(p, q) with coefficients in a finite dimensional representation and discuss their automorphic and geometric properties. In particular, these classes are generalizations of previous work of Kudla and Millson and give rise to Poincare dual forms for certain special cycles with non-trivial coefficients in arithmetic quotients of the associated symmetric space for the orthogonal group. Furthermore, we determine the behavior of these classes at the boundary of the Borel-Serre compactification of the associated locally symmetric space. As a consequence we are able to obtain new non-vanishing results for the special cycles. This is joint work with John Millson.

Speaker: *Jayce Getz* (Princeton University)
Title: Twisted relative trace formulae with a view towards unitary groups
Abstract: (joint work with E. Wambach) We introduce twisted relative trace formulae and provide tools to relate them to relative trace formulae modeled on the relative trace formula introduced by Jacquet and Lai. As an application, we consider the analogue for odd rank unitary groups of the work of Harder, Langlands and Rapoport on modular curves embedded in Hilbert modular surfaces.

Speaker: *Harald Grobner* (University of Vienna)
Title: Regular and residual Eisenstein cohomology classes for inner forms of symplectic groups
Abstract: Let G be a semisimple Lie group with maximal compact subgroup K and arithmetic congruence subgroup Γ. The locally symmetric space $\Gamma\backslash G/K$ carries interesting arithmetic information encoded in its cohomology groups $H^*(\Gamma\backslash G/K)$. One of the major tasks in understanding this cohomology is to understand a certain subspace $H^*_{Eis}(\Gamma\backslash G/K)$, called Eisenstein cohomology. We will focus on the case of inner forms of the symplectic group Sp_n and try to give an overview of interesting phenomena which one encounters in the search of a description of $H^*_{Eis}(\Gamma\backslash G/K)$. If time permits, we will present completely new results.

Speaker: *Lizhen Ji* (University of Michigan)
Title: Borel extension theorem and Mostow strong rigidity
Abstract: An important result for the Baily-Borel compactification of Hermitian locally symmetric spaces is the Borel extension theorem for holomorphic maps from the punctured disk into the Hermitian locally symmetric spaces. We discuss an analogue for the Deligne-Mumford compactification for the moduli space of curves. Then we will discuss the Mostow strong rigidity for mapping class groups of surfaces (or equivalently the moduli spaces of curves) and outer automorphism groups of free groups. We will also discuss non-isomorphism results between three closely related groups: lattice subgroups of Lie groups, mapping class groups, and outer automorphisms of free groups.

Speaker: *Christian Kaiser* (Max Planck Institute for Mathematics, Bonn)
Title: Irreducibility of Galois representations associated to automorphic forms of multiplicative groups of skew fields over function fields
Abstract: This is work in progress. Not all details have been checked yet. Let D be a skew field of degree d over some global field of characteristic p. Under some ramification constraint on D, Lafforgue and Lau associated to an automorphic representation π of $D^\times$ a d-dimensional Galois representation $\sigma(\pi)$ with the same L-functions as π. If π is (almost everywhere) tempered it is conjectured that $\sigma(\pi)$ is irreducible. We prove this conjecture in some cases. The main tool, which may be of independent interest, is an uniformization theorem for the moduli space of $\mathcal{D}$-shtukas of rank one at a point $(0, \infty) \in X \times X$ with $\mathrm{inv}_0 D = \frac{1}{d}$ and $\mathrm{inv}_\infty D = -\frac{1}{d}$ by a product of two copies of Drinfeld's upper half space Ω^d. Using results of Boyer and Dat on the cohomology of Ω^d and its coverings one concludes for skew fields D which have invariant $\frac{1}{d}$ and -$\frac{1}{d}$ at some different places (and under some more constraints). Using simple cases of the Jacquet-Langlands conjecture one can reduce many cases to this special situation.

Speaker: *Toshiyuki Kobayashi* (University of Tokyo)
Title: Restriction of unitary representations of real reductive groups
Abstract: Branching problems ask how an irreducible representation of a group decomposes when restricted to its subgroup. Having an observation of bad features of branching problems in a general non-compact setting even for real reductive symmetric pairs, I will discuss how to find a nice setting for branching problems, and give an upper estimate on multiplicities. If time allows, some applications are also presented.

Speaker: *Bernhard Kroetz* (Max Planck Institute for Mathematics, Bonn)
Title: Globalization of Harish-Chandra modules
Abstract: We will explain short and simple proofs of the globalization theorems of Casselman-Wallach and Kashiwara-Schmid. The work is partly joined with Joseph Bernstein and partly with Henrik Schlichtkrull.

Speaker: *Enrico Leuzinger* (University of Karlsruhe)
Title: Reduction theory for arithmetic and mapping class groups
Abstract: Reduction theory is concerned with the construction of (coarse) fundamental domains for groups acting properly discontinuously. I will describe a remarkable analogy between (a version of) reduction theory for arithmetic groups and corresponding results for mapping class groups. As an application I will discuss asymptotic cones for locally symmetric spaces and moduli spaces.

Speaker: *Joachim Mahnkopf* (University of Vienna)
Title: Traces on Hecke algebras and p-adic families of modular forms
Abstract: We prove that any modular eigenform f of level $\Gamma_1(Np)$, finite slope and weight k_0 can be placed into a p-adic family of modular eigenforms f_k of the same level and slope and weight k varying over all natural numbers which are sufficiently close to k_0 in the p-adic sense. Here, the term p-adic family means that a p-adic congruence between two weights k and k_0 entails a certain p-adic congruence between the corresponding eigenforms f_k and f_{k_0}. We also prove that the dimension of the slope subspace of the space of modular forms of weight k does not depend on the weight as long as we consider weights k which are sufficiently close to each other in the p-adic sense. Both these statements are predicted by the Mazur-Gouvea conjecture, which has been proven by Coleman. Our proof of these statements, which is completely different from Coleman's proof, is based on a comparison of (topological) trace formulas.

Speaker: *Werner Mueller* (University of Bonn)
Title: The Arthur trace formula and spectral theory on locally symmetric spaces

Abstract: In this talk I will discuss various applications of the Arthur trace formula to spectral theory on locally symmetric spaces. I will also discuss analytic problems related to the trace formula which need to be settled for further applications.

Speaker: *Martin Olbrich* (University of Luxembourg)
Title: Extending the realm of Patterson's conjectures
Abstract: This talk is concerned with analysis and spectral geometry of certain rank 1 locally symmetric spaces of typically infinite volume, namely geometrically finite ones. Geometrically finite spaces without cusps correspond to convex co-compact discrete groups. In 1993, Patterson formulated two conjectures concerning convex co-compact groups acting on real hyperbolic spaces. The first conjecture relates invariant currents supported on the limit set to the cohomology of the locally symmetric space, while the second gives a description of the singularities of Selberg's zeta function in terms of objects supported on the limit set. Both conjectures are now proved for many cases. We shall discuss how the conjectures should be modified in order to make sense for arbitrary geometrically finite locally symmetric spaces of rank 1. Already the inclusion of noncompact spaces of finite volume into the picture is an interesting and nontrivial task.

Speaker: *Gopal Prasad* (University of Michigan)
Title: Lengths of closed geodesics and isospectral locally symmetric spaces
Abstract: I will give an exposition of my recent work with Andrei Rapinchuk in which we have introduced a new notion of "weak commensurabilityöf Zariski-dense subgroups. Weak commensurability of arithmetic subgroups of semi-simple Lie groups turns out to have very strong consequences. Weak commensurability is intimately related to the commensurability of the set of lengths of closed geodesics on, and isospectrality of, locally symmetric spaces of finite volume (and with nonpositive sectional curvatures). Using our results we are able to answer Marc Kac's famous question "Can one hear the shape of a drum?for compact arithmetic locally symmetric spaces. Our proofs use algebraic number theory, class field theory, and also some results and conjectures from transcendental number theory.

Speaker: *Andras Vasy* (Stanford University)
Title: Scattering theory on symmetric spaces
Abstract: I will explain how methods from N-body scattering can be used to analyze the resolvent of the Laplacian and spherical functions on globally symmetric spaces (joint work with Rafe Mazzeo), and ongoing work with Rafe Mazzeo and Werner Mueller to extend the framework to locally symmetric spaces.

Speaker: *Dan Yasaki* (University of Massachusetts)
Title: Spines for Q-rank 1 groups
Abstract: Let $X = \Gamma\backslash D$ be an arithmetic quotient of a symmetric space associated to a semisimple algebraic group defined over $\mathbb{Q}$ with $\mathbb{Q}$-rank n. A result of Borel and Serre gives the vanishing of the cohomology of X in the top n degrees. Thus one can hope to find a Γ-equivariant deformation retraction of D onto a set D^0 having codimension n. When such a set exists, it is called a spine and is useful for computing the cohomology of X. Spines have been explicitly computed in many examples, and the general existence is known for groups associated to self-adjoint homogeneous cones. I will describe my work on the existence of spines for groups of $\mathbb{Q}$-rank 1.

4 Mathematical outcomes

The new results which have been presented can be roughly described by the following groups of key words:

- globalization theorems á la Casselmann–Wallach, restriction of representations to subgroups, length spectrum of geodesics, scattering theory, discrete spectrum of the Laplace operator
- construction of cohomology classes for $\Gamma\backslash X$ by geometric and representation theoretical methods, new invariants for classical modular forms, retracts of $\Gamma\backslash X$, liftings and relative trace formula
- compactifications and rigidity, also for mapping class groups or groups of outer automorphisms of free groups
- p-adic modular forms, Galois representations

5 References

Most of the talks are reports on work in progress. The interested reader is encouraged to contact the speakers about preprints. Hence we refer here only to some papers which give a first view on the field.

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