Discontinuous Galerkin Methods for Partial Differential Equations

Bernardo Cockburn (University of Minnesota)
Dominik Schötzau (University of British Columbia)
Chi-Wang Shu (Brown University)

November 25–November 30, 2007

The purpose of this meeting was to bring together researchers in a wide variety of areas working on discontinuous Galerkin (DG) methods for partial differential equations, to investigate and identify problems of current interest and to exchange ideas and viewpoints on the most recent developments of these methods. There were 33 participants, mostly from American and Canadian universities, including students and postdoctoral fellows. The program of the workshop consisted of 28 half-hour talks.

1 Overview of the Field

The origins of discontinuous Galerkin methods can be traced back to the seventies where they were introduced as non-standard discretization techniques for the numerical approximation of linear transport equations [1]. A remarkable adavantage of the original DG method is that the approximate solutions can be computed element-by-element when the elements are suitably ordered along the characteristic directions of the transport field. The success of DG methods for linear equations prompted several researchers to try to extend them to non-linear hyperbolic conservation laws. In the early eighties and beginning of the nineties, Cockburn and Shu introduced the Runge-Kutta discontinuous Galerkin (RKDG) methods for scalar conservation laws, see the review article [3] and the references therein. These methods are based on piecewise polynomial space discretizations, combined with total variation diminishing (TVD) explicit time-stepping algorithms. The resulting schemes have several important advantages as compared to, e.g., finite difference methods. The variational structure of DG methods greatly facilitates the handling of complicated geometries and elements of various shapes and types, as well as the treatment of boundary conditions. Moreover, DG mass matrices are block-diagonal and can be inverted at a very low computational cost, giving rise to very efficient time-stepping algorithms. Soon after their introduction, RKDG methods were extended to non-linear hyperbolic systems and to more general convection-diffusions problems.

2 Recent Developments

The nineties marked the start of a new era in the field of discontinuous Galerkin methods. The methods started to find their way into the main stream of computational fluid mechanics and began to be applied to a variety of problems for which they were not originally designed. Indeed, nowadays DG methods are successfully used in fields as diverse as meteorology, weather-forecasting, oceanography, gas dynamics, aero-acoustics, turbomachinery, turbulent flows, granular flows, oil recovery simulation, modeling of shallow water, transport of contaminants in porous media, viscoelastic flows, semi-conductor device simulation, incompressible fluid flow, structural mechanics, magnetohydrodynamics, and electromagnetism, among many others. The recent development of DG methods has been accompanied by a sharp increase of publications and workshops in the field. We refer the reader to [2] for the proceedings of the First International Symposium on Discontinuous Galerkin Methods, which took place in 1999 in Newport (Rhode Island), and to [4, 5] for two recent journals devoting special issues on discontinuous Galerkin methods.

Perhaps the main reason for this remarkable development is the fact that discontinuous Galerkin methods are extremely versatile and flexible; they combine elements from classical finite volume and finite element techniques. Their intrinsic stability properties make them extremely well-suited for problems where convection is dominant. Discontinuous Galerkin methods can deal robustly with partial differential equations of almost any kind, as well as with equations whose type changes within the computational domain. Therefore, they are naturally suited for multi-physics applications and for problems with highly varying material properties in complex geometries. Moreover, discontinuous Galerkin methods can easily handle irregularly refined meshes and variable approximation degrees. This property is referred to as hp-adaptivity, and is needed to resolve solution singularities efficiently and accurately.

The goal of the workshop was to discuss and identify the most relevant aspects of the current development of DG methods. One aspect of current interest is the proliferation of degrees of freedom in these methods. That is, on a fixed mesh, the number of degrees of freedom of a DG discretization might be substantially larger than the one of a standard finite element method of the same order. Hence, there is a great need for the development of efficient solvers and implementation techniques, as well as for the design of new methods that have fewer globally coupled degrees of freedom. Another aspect is the devising of DG methods for newly emerging applications in incompressible fluid flow, structural mechanics, electromagnetic wave propagation and in many other areas where convection plays a dominant role. A-posteriori estimation, error control and adaptivity are further aspects that are attracting a lot of attention. The numerical analysis of many adaptive algorithms is only in its beginnings, and many developments can be expected there over the next few years.

3 Talks

In the meeting, 28 half-hour lectures were given by the world's leading experts in the field of discontinuous Galerkin methods. The abstracts of the talks are listed below. (They were written by the speakers themselves.)

Antonietti, Paoloa (University of Nottingham)
Preconditioning discontinuous Galerkin approximations of elliptic problems
In recent years, much attention has been given to domain decomposition methods for linear elliptic problems that are based on a partitioning of the domain of the physical problem. Since the subdomains can be handled independently, such methods are very attractive for coarse-grain parallel computers. In this talk we shall present in a unified framework some non-overlapping additive and multiplicative Schwarz domain decomposition methods for the solution of the algebraic linear systems of equations arising from discontinuous Galerkin approximations of elliptic problems. In particular, two-level methods for both symmetric and non-symmetric DG schemes will be introduced and some interesting features, which have no analog in the conforming case, will be discussed. For symmetric DG discretizations, optimal convergence estimates will be presented, and we will show that the proposed Schwarz methods can be successfully accelerated with suitable Krylov iterative solvers. A discussion on the issue of preconditioning non-symmetric DG approximations of second order elliptic equations will be included. Numerical experiments to validate our theory and to illustrate the performance and robustness of the proposed two-level methods will be shown. This work has been carried out in collaboration with Blanca Ayuso (Departamento de Matemàticas, Universidad Autònoma de Madrid, Spain).

Ayuso, Blanca (University of Madrid)
Discontinuous Galerkin methods for advection-diffusion-reaction problems
We apply the weighted-residual approach recently introduced by Brezzi et. al., to derive discontinuous Galerkin formulations for advection-diffusion-reaction problems. We devise the basic ingredients to ensure stability and optimal error estimates in suitable norms, and propose two new methods. The talk is based on joint work with L. Donatella Marini from the University of Pavia.

Brenner, Susanne (Louisiana State University)
Multigrid algorithms for weakly over-penalized interior penalty methods
In this talk we will introduce two weakly over-penalized interior penalty methods that are stable for any penalty parameter and are quasi-optimal in both the energy norm and the L2 norm. We will also discuss multigrid algorithms for these interior penalty methods. Both theoretical and numerical results will be presented.

Celiker, Fatih (Wayne State University)
Adaptive stabilization of discontinuous Galerkin methods for nonlinear elasticity
We introduce a novel approach to stabilizing discontinuous Galerkin methods in nonlinear elasticity problems. The new stabilization strategy possesses the distinguishing feature of allowing the size of the stabilization term to vary throughout the mesh, and to automatically adjust the local level of stabilization according to the solution sought. This stabilization strategy is hence adaptive. The proposed scheme computationally efficient and remains stable for a fairly lengthy quasistatic loading path. This is demonstrated with two and three dimensional numerical examples. We further propose a slight modification of this approach for which we are able to prove theoretical estimates for the minimal values of the stabilization parameters defining the method.

Chen, Yanlai (University of Minnesota)
An adaptive high order discontinuous Galerkin method with error control for the Hamilton-Jacobi equations
We propose and study an adaptive version of the discontinuous Galerkin method for Hamilton-Jacobi equations. It works as follows. Given the tolerance and the degree of the polynomial of the approximate solution, the adaptive algorithm finds a mesh on which the approximate solution has an L^∞-distance to the viscosity solution no bigger than the prescribed tolerance. The algorithm uses three main tools. The first is an iterative solver combining the explicit Runge-Kutta Discontinuous Galerkin method and the implicit Newton's method that enables us to solve the Hamilton-Jacobi equations efficiently. The second is a new a posteriori error estimate based on the approximate resolution of an approximate problem for the actual error. The third is a method that allows us to find a new mesh as a function of the old mesh and the ratio of the a posteriori error estimate to the tolerance. We display extensive numerical evidence that indicates that, for any given polynomial degree, the method achieves its goal with optimal complexity independently of the tolerance. This is done in the framework of one-dimensional steady-state model problems with periodic boundary conditions.

Cheng, Yingda (University of Texas)
Discontinuous Galerkin solver for Boltzmann-Poisson transients
We present results of a discontinuous Galerkin scheme applied to deterministic computations of the transients for the Boltzmann-Poisson system describing electron transport in semiconductor devices. The collisional term models optical-phonon interactions which become dominant under strong energetic conditions corresponding to nano-scale active regions under applied bias. The proposed numerical technique is a finite element method using discontinuous piecewise polynomials as basis functions on unstructured meshes. It is applied to simulate hot electron transport in bulk silicon, in a silicon n^+-n-n^+ diode and in a double gated 12nm MOSFET. Additionally, the obtained results are compared to those of a high order WENO scheme simulation.

Dawson, Clint (University of Texas)
Good and bad aspects of discontinuous Galerkin methods for some geoscience problems
We will discuss the development and implementation of DG methods for two applications arising in geosciences. The first application is the shallow water equations. These equations are a hyperbolic/parabolic system of equations which is well-suited for approximation by the Runge-Kutta DG methods of Cockburn and Shu. We will describe recent successes and lessons learned in the application of hp-adaptive DG methods for shallow water under very complex flow scenarios. The second application is to flow in porous media, in particular flow in the vadose zone described by Richards' equation. Richards' equation is a highly nonlinear parabolic equation and requires implicit time discretization, which leads to the solution of large nonlinear and linear systems of equations. For these problems, DG methods have been observed to give solutions which are comparable to standard finite element methods, but are much more computationally intensive. Several numerical examples will be presented to illustrate our experiences in this area.

Dong, Bo (Brown University)
Optimal convergence of the original DG method for the transport-reaction equation on special meshes
We show that the approximation given by the original discontinuous Galerkin method for the transport-reaction equation in d space dimensions is optimal provided the meshes are suitably chosen: the L^2-norm of the error is of order $k + 1$ when the method uses polynomials of degree k. These meshes are not necessarily conforming and do not satisfy any uniformity condition; they are only required to be made of simplexes each of which has a unique *outflow* face. We also find a new, element-by-element postprocessing of the derivative in the direction of the flow which superconverges with order $k + 1$.

Gopalakrishnan, Jay (University of Floriday)
Hybridized DG methods
Discontinuous Galerkin methods have often been criticized for having too many unknowns. However, we show that DG methods can be made competitive with mixed and continuous Galerkin (CG) methods via hybridization. We achieve this by extending hybridization techniques already developed for mixed methods into a unified framework that allows hybridization of a variety of methods. This framework facilitates the discovery of new methods as well as new connections between DG, mixed and CG methods. Moreover, the unified framework allows easy coupling of these different methods, even across non-matching mesh interfaces.

Grote, Marcus (University of Basel)
Discontinuous Galerkin methods and local time stepping for second-order wave equations
The accurate and reliable simulation of wave phenomena is of fundamental importance in a wide range of engineering applications such as fiber optics, wireless communication, radar and sonar technology, and non-invasive testing. To address the wide range of difficulties involved, we consider symmetric interior penalty discontinuous Galerkin (IP-DG) methods, which easily handle elements of various types and shapes, irregular non-matching grids, and even locally varying polynomial order. Moreover, in contrast to standard (conforming) finite element methods, IP-DG methods yield an essentially diagonal mass matrix; hence, when coupled with explicit time integration, the overall numerical scheme remains truly explicit in time. To circumvent the severe stability (CFL) condition imposed on the time step by the smallest elements in the mesh, we propose local time-stepping schemes, which allow arbitrarily small time steps where small elements in the mesh are located. When combined with the symmetric IP-DG discretization, the resulting fully discrete scheme is explicit and exactly conserves a discrete energy. Starting from the standard second order “leap-frog” scheme, time integrators of arbitrary order of accuracy are derived. Numerical experiments illustrate the usefulness of these methods and validate the theory.

Guzmán, Johnny (University of Minnesota)
Superconvergent discontinuous Galerkin methods for second-order elliptic problems
We identify discontinuous Galerkin methods for second-order elliptic problems having superconvergence properties similar to those of the Raviart-Thomas and the Brezzi-Douglas-Marini mixed methods. These methods use polynomials of degree k for both the potential as well as the flux. We show that the approximate flux converges with the optimal order of $k + 1$, and that the approximate potential and its numerical trace superconverge, to suitably chosen projections of the potential, with order $k + 2$. We also apply element-by-element postprocessings of the approximate solution to obtain new approximations of the flux and the potential. The new approximate flux is proven to have normal components continuous across inter-element boundaries, to convergence with order $k + 1$, and to have a divergence converging in with order $k + 1$. The new approximate potential is proven to converge with order $k+2$. This is joint work with Bernardo Cockburn and Haiying Wang.

Hesthaven, Jan (Brown University)
Nodal DG-FEM for the modeling of free surface flows using high-order Boussinesq approximations
We shall discuss the modeling of free surface flows and fluid-structure interactions using high-order Boussinesq approximations. These sets of equations are characterized by being purely dispersive and strongly non-linear, with additional complications introduced by high-order spatial derivatives. We shall discuss the key elements of the formulation and some of the properties of the Boussinesq system. These properties shall be used to argue why DG-FEM may be a suitable approach for the solution of these equations. We shall continue to develop the basic elements required for solving this system, discussing a number of subtleties

and addressing practical concerns of performance and efficient solvers. The computational approach will be extensively validated with both benchmark tests and experimental data. This is work done in collaboration with A.P. Engsig-Karup (DTU, Denmark), P. Madsen (DTU, Denmark), H. Bingham (DTU, Denmark) and T. Warburton (Rice).

Kanschat, Guido (Texas A&M University)
Convergent adaptive algorithms for the interior penalty method
We discuss an adaptive strategy producing a sequence of meshes with a guaranteed error reduction in each step. Mesh refinement is based on a bulk criterion employing an energy norm a posteriori error estimate. The convergence proof relies on a modified discrete local efficiency. Numerical experiments show the feasibility of our approach.

Li, Fengyan (Rensselaer Polytechnic Institute)
A second order DGM based fast sweeping method for eikonal equations
The original fast sweeping method for solving static Hamilton-Jacobi equations is in the finite difference framework. By incorporating the causality of the equation into the discretization and combining the Gauss-Seidel iteration with alternating sweeping orderings, the method defines an efficient solver for these nonlinear equations with linear computational complexity. That is, the number of iterations needed for the error to be reduced to certain threshold is independent of the number of unknowns. On the other hand, this finite difference based fast sweeping method is only first order accurate. In this talk, I will discuss our recent progress in developing a second order fast sweeping method based on the discontinuous Galerkin discretization for an important family of static Hamilton-Jacobi equations - Eikonal equation. The main new components in our algorithm are the properly chosen numerical Hamiltonian in the DG formulation and the procedure to consistently enforce the causality. Numerical examples will be presented to demonstrate the performance of the proposed algorithm.

Peraire, Jaume (MIT)
A discontinuous Galerkin formulation for Lagrangian dynamic analysis
Dynamic analyses in solid mechanics are typically carried out by integrating the second order system expressing the balance of momentum. In this work, we follow a different approach. The non-linear governing equations are first cast as a system of first order conservation laws for momentum (and energy, if required by the constitutive law). In addition, a first order time evolution equation is also written for the deformation gradient tensor, $\mathbf{F}$. This results in a coupled system of 6 first order equations in 2 dimensions and 12 first order equations in 3 dimensions. The resulting system is found to possess involutions for the form $\nabla \times \mathbf{F} = \mathbf{0}$, which must be accounted for in the solution process in order to preserve stability. We develop an approximate Riemman solver and discretize the resulting equations using a high order discontinuous Galerkin method and integrate in time using an explicit Runge-Kutta algorithm. The method preserves momentum exactly and numerical results show excellent energy conservation properties for long time integrations. It is well known that for non-dissipative materials the solutions may develop shock waves which need to be handled numerically by the explicit addition of artificial viscosity. Shock capturing techniques developed in the context of non-linear conservation laws are then readily applicable. Unlike other formulations, the method is purely Lagrangian and therefore is formulated in the reference configuration. We also consider an alternative formulation in which the deformation gradient tensor is eliminated form the equations at the expense of introducing second order spatial derivatives. These are treated in the discretization using the CDG method. This formulation has a significantly reduced number of unknowns (4 in 2D and 6 in 3D) but seems to be less adequate to deal with strong discontinuities. This is work done in collaboration with P.-O. Persson (MIT) and J. Bonet (Swansea).

Persson, Per-Olof (MIT)
Preconditioning of Newton-GMRES solvers for discontinuous Galerkin problems
A fundamental problem with Discontinuous Galerkin methods is their high computational and storage cost. This is partly because they require more degrees of freedom than other methods, but mainly due to the wide nodal stencils which result in very large Jacobian matrices in implicit solvers. In this work we address the high cost of solving the corresponding linear systems of equations and propose a new preconditioner for implicit solution of stationary or time dependent Discontinuous Galerkin problems. The viscous terms are discretized using the Compact Discontinuous Galerkin (CDG) method, but the results should be representative for other schemes as well. We consider several existing preconditioners such as block-Jacobi and Gauss-Seidel combined with multi-level schemes which have been developed and tested for specific applications. While our

results are consistent with the claims reported, we find that these preconditioners lack robustness when used in more challenging situations involving low Mach numbers, stretched grids or high Reynolds number turbulent flows. We propose a preconditioner based on a coarse scale correction with post-smoothing based on a block incomplete LU factorization with zero fill-in (ILU0) of the Jacobian matrix. Our block-Minimum Discarded Fill algorithm numbers the elements such that the error in the ILU0 factorization is minimized in a greedy fashion, a step which turns out to be critical for convection dominated problems. While little can be said in the way of theoretical results, the proposed preconditioner is shown to perform remarkably well for a broad range of representative test problems. These include compressible flows ranging from very low Reynolds numbers to fully turbulent flows using the Reynolds Averaged Navier Stokes equations discretized on highly stretched grids. This is joint work with J. Peraire (MIT).

Rivière, Béatrice (University of Pittsburgh)
Coupling of surface and subsurface flows
The study of surface/subsurface interaction is important in the environmental problem of groundwater contamination. A mathematical weak formulation of the coupled Darcy flow with Navier-Stokes flow is presented. A numerical approach coupling discontinuous Galerkin and continuous finite element methods is analyzed. Extensions to time-dependent problem are given.

Ryan, Jennifer K. (Virginia Tech)
Local Post-processing for discontinuous Galerkin methods: Mesh, derivatives, and applications
In this presentation an overview of aspects of post-processing for discontinuous Galerkin methods will be given. Specifically, we will examine extensions to non-uniform meshes, derivative calculation, and applications to streamlines. Improving the accuracy in the solution and its' derivatives is important for many scientific applications in such areas as fluid mechanics and chemistry. The specific technique that we will examine was shown to improve the order of accuracy from k+1 of the DG approximation to 2k+1 for the post-processed solution over a uniform mesh where k is the highest degree polynomial used in the approximation. We will explore two techniques for extending the applications to smoothly varying and non-uniform mesh structures. Additionally, we will focus on extension of the existing kernel to improve the accuracy in the derivatives of the numerical solution. Lastly, we will present an application that uses one-dimensional filtering to aid in calculation of a streamline, regardless of the dimensionality of the solution.

Sherwin, Spencer J. (Imperial College London)
DG Methods using spectral/hp element discretisations: Balancing utility and efficiency
Given the growing interest in discontinuous Galerkin methods from both the academic and university sectors, it is of interest to ask why and when it is useful to apply the DG formulation in the context of high-order spectral/hp element discretisations. The spectral/hp element discretisation methodology originated from the confluence of two family of high order finite element methods: p-type finite elements and spectral element methods. Mimicking the p-type finite element techniques, spectral/hp elements employ hierarchical polynomial functions as trial and test functions as opposed to nodal Lagrange polynomials that are traditionally used in spectral element methods. A potential draw back of hierarchical expansions is that they lead to non-diagonal mass matrix systems which can be costly to invert in hyperbolic problem discretised explicitly in time. The discontinuous Galerkin method provided a significant advantage for this problem over the standard continuous Galerkin formulation particularly when using upwind fluxes which advantageously penalize the dispersive high frequencies. When we consider, however, parabolic or elliptic problems, the advantages of the DG formulation are not so apparent. For instance, the DG formulation is arguable more complicated to implement since it encapsulates many (but not all) of the continuous Galerkin components. DG formulations typically lead to a greater number of degrees of freedom compared to classical Galerkin formulations, and do not necessarily provide significant improvement in accuracy or compactness, at least for problems with constant coefficients. However, the ability to adapt a DG formulation and the absence of a direct stiffness (or global assembly) process can lead to greater flexibility/adaptivity. In addition, if the solution is required to exist in a discontinuous space then there may not be many obvious alternative. In summary if one already has a continuous Galerkin implementation the benefits of converting to a DG formulation are questionable for many standard problems, especially when using low order polynomial spatial approximations. In this presentation we will explore the above competing arguments. Finally, we will also outline some of the potential benefits of the recently proposed LDG-Hybrid approach where the issue of additional degrees of freedom

have been addressed and which may offer additional advantages in massively parallel implementations. This is joint work with Robert M. Kirby (University of Utah).

Shu, Chi-Wang (Brown University)
L^2-Stability analysis of the central discontinuous Galerkin method and a comparison between the central and regular discontinuous Galerkin methods
We give stability analysis and error estimates for the recently introduced central discontinuous Galerkin method when applied to linear hyperbolic equations. A comparison between the central discontinuous Galerkin method and the regular discontinuous Galerkin method in this context is also made. Numerical experiments are provided to validate the quantitative conclusions from the analysis. This is joint work with Yingjie Liu, Eitan Tadmor and Mengping Zhang.

Van der Vegt, Jaap (University of Twente)
Discontinuous Galerkin finite element methods for hyperbolic nonconservative partial differential equations
In this presentation a new space-time discontinuous Galerkin finite element (DGFEM) formulation will be discussed for partial differential equations containing nonconservative products, such as occur in dispersed multiphase flow equations. Standard DGFEM formulations cannot be applied to nonconservative partial differential equations. We therefore introduce the theory of weak solutions for nonconservative products into the DGFEM formulation leading to the new question how to define the path connecting left and right states across a discontinuity. The effect of different paths on the numerical solution is investigated and found to be small. We also introduce a new numerical flux that is able to deal with nonconservative products. The scheme is applied to two different systems of partial differential equations. First, we consider the shallow water equations, where topography leads to nonconservative products. Second, a simplification of a depth-averaged two-phase flow model is discussed. This model contains more intrinsic nonconservative products. This is joint work Sander Rhebergen and Onno Bokhove.

Wang, Wei (Brown University)
The discontinuous Galerkin method for the multiscale modeling of dynamics of crystalline solids
We present a multiscale model for numerical simulation of dynamics of crystalline solids. The method couples nonlinear elastodynamics as the continuum description and molecular dynamics as another component at the atomic scale. The governing equations on the macroscale are solved by the discontinuous Galerkin method, which is built up with an appropriate local curl-free space to produce coherent displacement field. The constitutive data are based on the underlying atomistic model: it is either calibrated prior to the computation or obtained from molecular dynamics as the computation proceeds. The decision to use either the former or the latter is made locally for each cell based on suitable criteria. This is joint work with Xiantao Li (Penn State) and Chi-Wang Shu (Brown University).

Warburton, Timothy (Rice University)
Advances in wave propagation with the discontinuous Galerkin method
A range of important features relating to the practical application of discontinuous Galerkin method for wave propagation will be discussed. Recent investigations of the spectral properties of the discrete discontinuous Galerkin operators have revealed important connections with their continuous Galerkin counter parts. Theoretical and numerical results will be shown which demonstrate the correct asymptotic behavior of these methods and controls spurious solutions under mild assumptions. Given the suitability of DG for solving Maxwell's equations and their ability to propagate waves over long distance, it is natural to seek effective boundary treatments for artificial radiation boundary conditions. A new family of far field boundary conditions will be introduced which gracefully transmit propagating and evanescent components out of the domain. These conditions are specifically formulated with DG discretizations in mind, however they are also relevant for a range of numerical methods. There is an Achilles heel to high order discontinuous Galerkin methods when applied to conservation laws. The methods are typically constructed with polynomial field representations and unfortunately these suffer from excess maximum gradients near the edges of elements. I will describe a simple filtering process that allows us to reduce these anomalous gradients and provably yield a dramatic increase in the maximum allowable time step. Finally, I will discuss progress in using a posteriori error estimates for mesh adaptivity and demonstrate guaranteed error reduction on refinement for some model (static) problems. These results indicate a need for very fine local refinement of meshes to accurately capture solution singularities. I will show a very simple approach for local time stepping with discontinuous Galerkin methods in order to practically use such meshes in time-domain computations.

Wheeler, Mary (University of Texas)

Coupling discontinuous Galerkin and mixed finite element discretizations using mortar elements

Discontinuous Galerkin and mixed finite element (MFE) methods are two popular methods that possess local mass conservation. In this paper we investigate DG-DG and DG-MFE domain decomposition couplings using mortar finite elements to impose weak continuity of fluxes and pressures on the interface. The subdomain grids need not match and the mortar grid may be much coarser, giving a two-scale method. Convergence results in terms of the fine subdomain scale h and the coarse mortar scale H are established for both types of couplings. In addition, a non-overlapping parallel domain decomposition algorithm is developed, which reduces the coupled system to an interface mortar problem. The properties of the interface operator are analyzed. Computational results are presented.

Wihler, Thomas P. (McGill University)

A-posteriori error estimation for hp-DG time-stepping for parabolic PDEs

We will present an hp-version a posteriori error analysis for the time discretization of parabolic problems by the discontinuous Galerkin time-stepping. The resulting error estimators are fully explicit with respect to the local time steps and approximation orders. Their performance within an hp-adaptive time stepping procedure shall be illustrated with a series of numerical experiments. This talk comprises joint work with Dominik Schötzau (University of British Columbia).

Xu, Yan (University of Science and Technology of China)

A local discontinuous Galerkin method for the Camassa-Holm equation

In this paper, we develop, analyze and test a local discontinuous Galerkin (LDG) method for solving the Camassa-Holm equation which contains nonlinear high order derivatives. The LDG method has the flexibility for arbitrary h and p adaptivity. We prove the L^2 stability for general solutions and give a detailed error estimate for smooth solutions, and provide numerical simulation results for different types of solutions of the nonlinear Camassa-Holm equation to illustrate the accuracy and capability of the LDG method. This is joint work with Chi-Wang Shu (Brown University).

Zhang, Yongtao (University of Notre Dame)

Application of a discontinuous Galerkin finite element method to reaction-diffusion systems in developmental biology

Nonlinear reaction-diffusion systems which arise from mathematical modeling in developmental biology are usually highly stiff in both diffusion and reaction terms, and they are built on high dimensional complex geometrical domains due to the complex shape of embryos. Computational challenges come from both the temporal direction and the spatial direction. The stiffness of these reaction-diffusion equations demands efficient temporal numerical discretization. Complex geometrical domain can be handled by finite element spatial discretization. Based on the recent work by Cheng and Shu, we combined the DG scheme with Strang's operator splitting and the Crank-Nicholson discretization to solve the nonlinear reaction-diffusion systems, and overcome the difficulties from high stiffness of the system and the complexity of the geometrical domain. Directly solving a coupled nonlinear system like the standard implicit schemes was avoided. We applied the method to various reaction-diffusion models in developmental biology, including a system from the skeletal pattern formation in developing chick limb, to show the accuracy and efficiency of the method. This is a joint work with Jianfeng Zhu, Mark Alber and Stuart A. Newman.

Zhang, Zhimin (Wayne State University)

Discontinuous Galerkin methods for singularly perturbed problems

We study DG methods for convection diffusion equations. When the diffusion coefficient ϵ is small, the underling equation is singularly perturbed. Compared with continuous Galerkin methods, i.e., the traditional finite element methods, DG methods have the advantage to avoid non-physical oscillations. Our main concern is the convergence independent of the singular perturbation parameter ϵ. Furthermore, we shall apply recovery techniques to the DG methods by identifying some superconvergence points and using them to reconstruct local solutions in achieving higher order convergence on the whole domain or some sub-domains. Both the one-dimensional and two-dimensional model problems are discussed. Numerical examples will be presented. This is joint work with Huiqing Zhu (Wayne State).

4 Outcome of the Meeting

In the meeting, the most pressing and relevant issues of the current development of discontinuous Galerkin methods were discussed and identified: the devising of new methods with fewer degrees of freedom, the devising of new methods for fluid mechanics, solid mechanics, electromagnetism, and other applications of engineering practise, the analysis of new adaptive algorithms, and the development of new preconditioning and solution techniques.

Since the workshop concentrated on discontinuous Galerkin methods, it was quite efficient in terms of exchange and presentation of ideas. On the other hand, the invited researchers represented quite different directions of research so that quite different aspects and viewpoints on discontinuous Galerkin methods were presented. Our mix of junior and senior researchers created an atmosphere that was very inspiring and stimulating for all the participants. We received many positive comments from the participants about the quality of the talks, the facilities and how the workshop was run. As a result of this, Chi-Wang Shu announced the organization of a special issue on DG methods in the Journal of Scientific Computing, with contributions from the researchers invited to the workshop.

Finally, we would like to thank the staff at BIRS for their help and hard work to make this a very enjoyable workshop.

References

[1] B. Cockburn, G.E. Karniadakis, and C.-W. Shu. The development of discontinuous Galerkin methods. In B. Cockburn, G.E. Karniadakis, and C.-W. Shu, editors, *Discontinuous Galerkin Methods: Theory, Computation and Applications*, volume 11 of *Lect. Notes Comput. Sci. Eng.*, pages 3–50. Springer Verlag, 2000.

[2] B. Cockburn, G.E. Karniadakis, and C.-W. Shu, editors. *Discontinuous Galerkin Methods. Theory, Computation and Applications*, volume 11 of *Lect. Notes Comput. Sci. Engrg.* Springer Verlag, 2000.

[3] B. Cockburn and C.-W. Shu. Runge–Kutta discontinuous Galerkin methods for convection–dominated problems. *J. Sci. Comp.*, 16:173–261, 2001.

[4] B. Cockburn and C.-W. Shu. Foreword for the special issue on discontinuous Galerkin methods. *J. Sci. Comp.*, 22:1–3, 2005.

[5] C. Dawson. Foreword for the special issue on discontinuous Galerkin methods. *Comput. Methods Appl. Mech. Engrg.*, 195:3, 2006.

List of Participants

Antonietti, Paola (University of Nottingham)
Ayuso, Blanca (University of Madrid)
Brenner, Susanne (Louisiana State University)
Celiker, Fatih (Wayne State University)
Chen, Yanlai (University of Minnesota)
Cheng, Yingda (University of Texas)
Cockburn, Bernardo (University of Minnesota)
Dawson, Clint (University of Texas)
Djoumna, Georges (University of Laval)
Dong, Bo (Brown University)
Gopalakrishnan, Jay (University of Floriday)
Grote, Marcus (University of Basel)
Guzmán, Johnny (University of Minnesota)
Hesthaven, Jan (Brown University)
Kanschat, Guido (Texas A&M University)
Li, Fengyan (Rensselaer Polytechnic Institute)

Peraire, Jaume (MIT)
Persson, Per-Olof (MIT)
Rivière, Beatrice (University of Pittsburgh)
Ryan, Jennifer K. (Virginia Tech)
Schötzau, Dominik (University of British Columbia)
Sherwin, Spencer J. (Imperial College London)
Shu, Chi-Wang (Brown University)
Van der Vegt, Jaap (University of Twente)
Wang, Wei (Brown University)
Warburton, Timothy (Rice University)
Wei, Xiaoxi (University of British Columbia)
Wheeler, Mary (University of Texas)
Wihler, Thomas P. (McGill University)
Xu, Yan (University of Science and Technology of China)
Zhang, Yongtao (University of Notre Dame)
Zhang, Zhimin (Wayne State University)
Zhu, Liang (University of British Columbia)