# A SURVEY OF COMPLETELY BOUNDED MAPS

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Abstract. These notes are an expanded version of my talk at the conference on "Operator Structures in Quantum Information Theory" at the Banff International Research Station on February 13, 2007. They are intended to give an introduction to some of the topics in the theory of operator spaces and completely bounded maps that might be useful for researchers in quantum information theory

## 1. Introduction and Key Concepts

Given a Hilbert space $\mathcal{H}$, and operators, $T_{i,j} \in B(\mathcal{H}), 1 \leq i \leq m, 1 \leq j \leq n$, we identify the $m \times n$ matrix of operators, $(T_{i,j})$ with an operator from $\mathcal{H}^{(n)} = \mathcal{H} \oplus \ldots \oplus \mathcal{H}$(n copies) to $\mathcal{H}^{(m)} = \mathcal{H} \oplus \ldots \mathcal{H}$(m copies) by regarding vectors in these spaces as columns and performing matrix multiplication. That is, we idenitfy $M_{m,n}(B(\mathcal{H})) \equiv B(\mathcal{H}^{(n)}, \mathcal{H}^{(m)})$. This endows $M_{m,n}(B(\mathcal{H}))$ with a norm and this collection of norms on $B(\mathcal{H})$ are often referred to as the **matrix norms** on $B(\mathcal{H})$.

**Definition 1.** *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{M} \subseteq B(\mathcal{H})$ be a subspace. Then the inclusion, $M_{m,n}(\mathcal{M}) \subseteq M_{m,n}(B(\mathcal{H}))$ endows this vector space with a collection of matrix norms and we call, $\mathcal{M}$, together with this collection of matrix norms on $M_{m,n}(\mathcal{M})$ a (concrete)* **operator space.** *When $m = n$, we set $M_n(\mathcal{M}) = M_{n,n}(\mathcal{M})$.*

Thus, an operator space carries not just an inherited norm structure, but these additional matrix norms.

It is easily checked, that if $\mathcal{A}$ is any C*-algebra and $\pi : \mathcal{A} \to B(\mathcal{H})$ is a one-to-one *-homomorphism(and hence an isometry), then the collection of norms on $M_{m,n}(\pi(\mathcal{A}))$ is independent of the particular representation $\pi$, and hence, the operator space structure of a C*-algebra is independent of the particular(faithful) representation. Hence, each subspace $\mathcal{M} \subseteq \mathcal{A}$ is also endowed with a particular collection of matrix norms and so we also refer to a subspace of a C*-algebra as an **operator space,** when we wish to emphasize its matrix norm structure.

**Definition 2.** *Given a C*-algebra $\mathcal{A}$, an operator space $\mathcal{M} \subseteq \mathcal{A}$, and a linear map, $\phi : \mathcal{M} \to B(\mathcal{H})$, we define $\phi_n : M_n(\mathcal{M}) \to M_n(B(\mathcal{H}))$ by*

2000 *Mathematics Subject Classification.* Primary 46L15; Secondary 47L25.

$\phi_n((a_{i,j})) = (\phi(a_{i,j}))$. *We call* $\phi$ **completely bounded**, *if*

$$\|\phi\|_{cb} \equiv \sup_n \|\phi_n\|,$$

*is finite.*

It is easily checked that $\|\phi_n\| \le \|\phi_{n+1}\|$ and that $\|\phi_n\| \le n\|\phi\|$ [3, Chapter 1].

Some authors prefer tensor notation for the above concepts. In the tensor notation, $M_n(\mathcal{M}) \equiv \mathcal{M} \otimes M_n$ and $\phi_n \equiv \phi \otimes id_n$, where $id_n : M_n \to M_n$ is the identity map.

**Definition 3.** *Given a C*-algebra* $\mathcal{A}$ *and* $\mathcal{M} \subseteq \mathcal{A}$ *an operator space, we set* $\mathcal{M}^* = \{a^* : a \in \mathcal{M}\}$, *which is another operator space. If* $\phi : \mathcal{M} \to B(\mathcal{H})$, *is a linear map, then we define* $\phi^* : \mathcal{M}^* \to B(\mathcal{H})$ *by* $\phi^*(b) = \phi(b^*)^*$, *which is another linear map.*

It is easily checked that $\|\phi_n\| = \|(\phi^*)_n\|$ and hence that $\|\phi\|_{cb} = \|\phi^*\|_{cb}$.

One note of caution. When $\mathcal{M} = M_n, B(\mathcal{H}) = M_k$, and $\phi : M_n \to M_k$ then these spaces are also Hilbert spaces and consequently there is also a **dual map,** which is more commonly used in quantum information theory. To avoid confusion, we denote the dual map by $\phi^\dagger : M_k \to M_n$. In particular, if $A \in M_{k,n}, B \in M_{n,k}$, and $\phi : M_n \to M_k$, is defined by $\phi(X) = AXB$, then $\phi^* : M_n \to M_k$ is given by $\phi^*(X) = B^*XA^*$, while $\phi^\dagger : M_k \to M_n$ is given by $\phi^\dagger(Y) = BYA$.

**Definition 4.** *If* $\mathcal{A}$ *is a unital C*-algebra, then a subspace* $\mathcal{S} \subseteq \mathcal{A}$ *such that* $1 \in \mathcal{S}$, *and* $\mathcal{S}^* = \mathcal{S}$ *is called an* **operator system.**

Thus, operator systems are operator spaces and have matrix norms. But the additional hypotheses guarantee that if we let $\mathcal{A}^+$ denote the positive elements of the C*-algebra, then $\mathcal{S}$ is the span of $\mathcal{S}^+ \equiv \mathcal{S} \cap \mathcal{A}^+$, which is a cone in $\mathcal{S}$. We also have that $M_n(\mathcal{S})$ is the span of $M_n(\mathcal{S})^+ = M_n(\mathcal{S}) \cap M_n(\mathcal{A})^+$. The vector spaces, $M_n(\mathcal{S})$ together with the cones $M_n(\mathcal{S})^+$ is often referred to as the **matrix ordering** on $\mathcal{S}$.

**Definition 5.** *Given a unital C*-algebra* $\mathcal{A}, \mathcal{S} \subseteq \mathcal{A}$ *and a map* $\phi : \mathcal{S} \to B(\mathcal{H})$, *we call* $\phi$ **completely positive,** *provided that* $\phi_n$ *is positive for all* $n$, *that is provided that* $(a_{i,j}) \in M_n(\mathcal{S})^+$ *implies that* $(\phi(a_{i,j})) \in M_n(B(\mathcal{H}))^+$.

For completely positive maps, one has that $\|\phi\|_{cb} = \|\phi\| = \|\phi(1)\|$ and that $\phi^* = \phi$.

The following objects allow one to relate much of the theory of completely bounded maps to the more familiar theory of completely positive maps.

**Definition 6.** *Let* $\mathcal{A}$ *be a unital C*-algebra and let* $\mathcal{M} \subseteq \mathcal{A}$ *be an operator space, then we define an operator system* $\mathcal{S}_\mathcal{M} \subseteq M_2(\mathcal{A})$, *by*

$$\mathcal{S}_\mathcal{M} \equiv \{ \begin{pmatrix} \lambda 1 & a \\ b^* & \mu 1 \end{pmatrix} : \lambda \in \mathbb{C}, \mu \in \mathbb{C}, a \in \mathcal{M}, b \in \mathcal{M} \}.$$

**Theorem 7.** [3] *Let $\mathcal{A}$ be a unital $C^*$-algebra, $\mathcal{M} \subseteq \mathcal{A}$ be an operator space and let $\phi : \mathcal{M} \to B(\mathcal{H})$ be linear. Then $\|\phi\|_{cb} \leq 1$ if and only if $\Phi : \mathcal{S}_{\mathcal{M}} :\to M_2(B(\mathcal{H}))$ is completely positive, where*

$$\Phi(\begin{pmatrix} \lambda 1 & a \\ b^* & \mu 1 \end{pmatrix}) = \begin{pmatrix} \lambda I_{\mathcal{H}} & \phi(a) \\ \phi(b)^* & \mu I_{\mathcal{H}} \end{pmatrix}.$$

In particular, using this identification of completely contractive maps with "corners" of unital completely positive maps, one extension theorem:

**Theorem 8** (Arveson's Extension Theorem)**.** [3] *Let $\mathcal{S} \subseteq \mathcal{A}$ be an operator system and let $\phi : \mathcal{S} \to B(\mathcal{H})$ be completely positive, then there exists a completely positive map $\psi : \mathcal{A} \to B(\mathcal{H})$ that extends $\phi$, i.e., such that $\psi(a) = \phi(a)$ for every $a \in \mathcal{S}$.*

quickly yields another:

**Theorem 9** (Wittstock's Extension Theorem)**.** [3] *Let $\mathcal{M} \subseteq \mathcal{A}$ be an operator space and let $\phi : \mathcal{M} \to B(\mathcal{H})$ be completely bounded, then there exists a completely bounded map $\psi : \mathcal{A} \to B(\mathcal{H})$ that extends $\phi$ and satisfies $\|\psi\|_{cb} = \|\phi\|_{cb}$.*

To obtain the second from the first, one first scales $\phi$ so that $\|\phi\|_{cb} = 1$, then applies Arveson's theorem to extend $\Phi : \mathcal{S}_{\mathcal{M}} \to M_2(B(\mathcal{H}))$, to $\Psi : M_2(\mathcal{A}) \to M_2(B(\mathcal{H}))$, and then lets $\psi$ be the corresponding (1,2)-corner of $\Psi$.

In a similar fashion, the representation theorem for completely positive maps:

**Theorem 10** (Stinespring's Representation Theorem)**.** [3] *Let $\mathcal{A}$ be a unital $C^*$-algebra and let $\phi : \mathcal{A} \to B(\mathcal{H})$ be a completely positive map, then there exists a Hilbert space $\mathcal{K}$, a bounded operator $V : \mathcal{H} \to \mathcal{K}$ and a unital *-homomorphism, $\pi : \mathcal{A} \to B(\mathcal{H})$ such that $\phi(a) = V^*\pi(a)V$, for every $a \in \mathcal{A}$.*

becomes:

**Theorem 11** (The Generalized Stinespring Theorem)**.** [3] *Let $\mathcal{A}$ be a unital $C^*$-algebra and let $\phi : \mathcal{A} \to B(\mathcal{H})$ be a completely bounded map, then there exists a Hilbert space $\mathcal{K}$, bounded operators $V : \mathcal{H} \to \mathcal{K}, W : \mathcal{H} \to \mathcal{K}$ and a unital *-homomorphism, $\pi : \mathcal{A} \to B(\mathcal{K})$, such that $\|\phi\|_{cb} = \|V\|\|W\|$ and $\phi(a) = V^*\pi(a)W$, for every $a \in \mathcal{A}$.*

Note that in Stinespring's theorem, we also have that $\|\phi\|_{cb} = \|\phi(1)\| = \|V^*V\| = \|V\|^2$.

The generalization of Stinespring's theorem to completely bounded maps yields the following "polar form" for completely bounded maps. To motivate this result note that for operators, if we set $|T| = \sqrt{T^*T}$, then $\begin{pmatrix} |T^*| & T \\ T^* & |T| \end{pmatrix}$ is positive.

**Corollary 12.** *Let $\mathcal{A}$ be a unital $C^*$-algebra and let $\phi : \mathcal{A} \to B(\mathcal{H})$ be completely bounded, then there exists completely positive maps, $\phi_1, \phi_2 : \mathcal{A} \to B(\mathcal{H})$, with $\|\phi_1(1)\| = \|\phi_2(1)\| = \|\phi\|_{cb}$, such that $\Phi : M_2(\mathcal{A}) \to M_2(B(\mathcal{H}))$ is completely positive, where*

$$\Phi(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} \phi_1(a) & \phi(b) \\ \phi^*(c) & \phi_2(d) \end{pmatrix}.$$

When $\|\phi\|_{cb} \le 1$, then $\phi_1, \phi_2$ can both be taken to be unital completely positive maps. Thus, the above corollary is another example of the meta-theorem that completely contractive maps are the "corners" of unital completely positive maps.

The generalized Stinespring theorem, unfortunately, has no good uniqueness criteria, unlike the usual Stinespring theorem. The dificulty stems from the fact that the two completely positive maps, $\phi_1, \phi_2$ are not uniquely determined by $\phi$. Generally, there are many possible extensions of the completely positive map $\Phi$ from the operator system $\mathcal{S}_{\mathcal{A}}$ to $M_2(\mathcal{A})$ and this allows for a great deal of non-uniqueness. Thus, in particular, the above "polar form" of a completely bounded map is not unique. Latter we will discuss an alternative to the generalized Stinespring that eliminates some, but not all, of this non-uniqueness.

Just as one obtains the Choi-Kraus representation of completely positive maps from $M_n$ to $M_k$ by specializing Stinespring's theorem to these algebras, one obtains a similar representation of completely bounded maps.

**Theorem 13** (Choi-Kraus Representation Theorem)**.** *Let $\phi : M_n \to M_k$ be completely positive, then there exists matrices, $A_i \in M_{n,k}, 1 \le i \le nk$, such that $\phi(X) = \sum_i A_i^* X A_i$.*

**Theorem 14** (CB Representation Theorem)**.** *Let $\phi : M_n \to M_k$, then there exists matrices, $A_i \in M_{k,n}, 1 \le i \le m$, and matrices $B_i \in M_{n,k}, 1 \le i \le m$, such that $\phi(X) = \sum_i A_i X B_i$, with $\|\phi\|_{cb}^2 = \|\sum_i A_i A_i^*\|\|\sum_i B_i^* B_i\|$ and $m \le nk$.*

We shall call *any* representation $\phi(X) = \sum_i A_i X B_i$ a **generalized Choi-Kraus representation.** Note that if we have any generalized Choi-Kraus representation of $\phi$ then,

$$\phi(X) = (A_1, \dots, A_m) \begin{pmatrix} X & 0 & \dots & 0 \\ 0 & X & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & X \end{pmatrix} \begin{pmatrix} B_1 \\ \vdots \\ B_m \end{pmatrix},$$

where the term in the middle represents the $m \times m$ block diagonal matrix each of whose blocks is $X$ and hence,

$$\|\phi\|_{cb} \le \|(A_1, ..., A_m)\|\|\begin{pmatrix} B_1 \\ \vdots \\ B_m \end{pmatrix}\| = \|\sum_i A_i A_i^*\|^{1/2} \|\sum_i B_i^* B_i\|^{1/2},$$

which explains the asymmetry in the roles of the A's and B's.

This also leads to the following:

**Corollary 15.** *Let $\phi : M_n \to M_k$, then*

$$\|\phi\|_{cb} = \inf\{\|\sum_i A_i A_i^*\|^{1/2} \|\sum_i B_i^* B_i\|^{1/2}\},$$

*where the infimum is taken over all generalized Choi-Kraus representations of $\phi$.*

Recall that $\phi(X) = \sum_i A_i^* X A_i$ is unital, completely positive if and only if $\phi^\dagger(Y) = \sum_i A_i Y A_i^*$ is trace-preserving, completely positive. This can be seen from the representation, since $\phi(I_n) = I_k$ if and only if $\sum A_i^* A_i = I_k$. My apologies for reversing the usual roles of $A_i$ and $A_i^*$, but this is one difference in notation between operator algebraists and the quantum information community.

Thus, by duality, we obtain the following theorem.

**Theorem 16.** *Let $\psi : M_k \to M_n$, then the following are equivalent:*

- *there exist matrices $A_i \in M_{k,n}, B_i \in M_{n.k}, 1 \le i \le nk$ such that $\psi(Y) = \sum_i B_i Y A_i$, with $\|\sum_i A_i A_i^*\| \|\sum_i B_i^* B_i\| \le 1$,*
- *there exists completely positive trace preserving maps, $\psi_1, \psi_2 : M_k \to M_n$, such that $\Psi : M_2(M_k) \to M_2(M_n)$ defined by $\Psi(\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}) = \begin{pmatrix} \psi_1(X) & \psi(Y) \\ \psi^*(Z) & \psi_2(W) \end{pmatrix}$ is completely positive(and trace-preserving),*
- *$\psi^\dagger : M_n \to M_k$ is completely contractive.*

## 2. Estimation and Computation of Completely Bounded Norms

In some recent results and conjectures in quantum information theory it has become necessary to compute or at least bound the cb-norm of various maps. Here we summarize a few key facts and present an algorithm for computing completely bounded norms that is useful in finite dimensions.

First it is useful to know that every map into a finite dimensional space is completely bounded.

**Theorem 17** (Smith)**.** [3] *Let $\mathcal{M}$ be an operator space and let $\psi : \mathcal{M} \to M_n$, then $\|\psi\|_{cb} = \|\psi_n\| \le n\|\psi\|$.*

For maps whose domain is $M_n$ a result of Haagerup shows that, in general, $\|\psi\|_{cb} \neq \|\psi_m\|$, no matter how large one takes $m$[3, p. 114], but we do have an upper bound. This result is not explicitly in the literature so we provide a proof below, that uses some concepts that we will introduce in Section 4 and, perhaps, illustrates their utility. For now, it is enough to know that given a finite dimensional normed space $X$, there exists a constant $\alpha(X)$ called the *alpha constant* of the space, with the property that

$$\|\phi\|_{cb} \le \alpha(X)\|\phi\|$$

for any map with domain an operator space that is isometrically isomorphic to $X$ as normed spaces. Given two finite dimensional normed spaces $X, Y$ of the same dimension one has

$$\alpha(X) \leq d(X,Y)\alpha(Y),$$

where $d(X,Y)$ denotes the *Banach-Mazur distance* between the spaces. These concepts and results can be found in [4].

**Theorem 18.** *Let $\mathcal{M}$ be an operator space and let $\phi : M_n \to \mathcal{M}$, then $\|\phi\|_{cb} \leq n\sqrt{n}\|\phi\|$.*

*Proof.* Let $\|X\|_2$, denote the Hilbert-Schmidt norm of a matrix, i.e., the norm when we identify $M_n$ with Euclidean space $\mathbb{C}^{n^2}$. Since $\|X\| \leq \|X\|_2 \leq \sqrt{n}\|X\|$, we have that the Banach-Mazur distance satisfies $d(M_n, \mathbb{C}^{n^2}) \leq \sqrt{n}$.

Hence, $\alpha(M_n) \leq d(M_n, \mathbb{C}^{n^2})\alpha(\mathbb{C}^{n^2}) \leq \sqrt{n}\alpha(\mathbb{C}^{n^2})$. Finally, it is shown in [4], that for Euclidean space, $\alpha(\mathbb{C}^m) \leq \sqrt{m}$, from which the result follows. □

Combining these results we have:

**Corollary 19.** *Let $\phi : M_n \to M_k$ be a linear map, then $\|\phi\|_{cb} \leq \min\{k, \sqrt{n^3}\}$.*

We now turn to the problem of actually computing the norm of a map $\phi : M_n \to M_k$. By the above results we know that to compute the cb-norm we need to do a minimization over all generalized Choi-Kraus representations. This turns out to be somewhat more attainable than might be imagined and we present an algorithm for computing the cb-norm of such maps.

We first describe the algorithm and then justify it later.

**An Algorithm for Computing the CB-Norm.**

We assume that we are given a map $\phi : M_n \to M_k$, some generalized Choi-Kraus representation $\phi(X) = \sum_{i=1}^m A_i X B_i$ and we wish to compute $\|\phi\|_{cb}$.

**Step 1.** Find a basis, $\{C_1, ..., C_l\}$ for the span of $\{B_1, ..., B_m\}$ and express $B_i = \sum d_{i,j} C_j$

**Step 2.** Using the expressions for each $B_i$ as a linear combination of $C_j$ we may re-write $\phi(X) = \sum_{j=1}^l D_j X C_j$. In fact, we have $\phi(X) = \sum_i A_i X(\sum_j d_{i,j} C_j) = \sum_j (\sum_i d_{i,j} A_i) X C_j$. Thus,

$$D_j = \sum_i d_{i,j} A_i.$$

**Step 3.** Find a basis $\{E_1, ..., E_p\}$ for the span of $\{D_1, ..., D_l\}$ express each $D_j$ as a linear combination and repeat Step 2, to obtain

$$\phi(X) = \sum_{i=1}^{m} E_i X F_i,$$

where the $F_i$'s are the corresponding linear combinations of the $C_j$'s.

Remarkably, at this stage it is a theorem that the sets $\{E_1, ..., E_p\}$ and $\{F_1, ..., F_p\}$ are linearly independent, so this process terminates!

**Step 4.** Given an invertible $S = (s_{i,j}) \in M_p$ with inverse $S^{-1} = (t_{i,j}) \in M_p$, let $H_i = \sum_j s_{i,j} F_j$, and $G_j = \sum_i t_{i,j} E_i$. Then

$$\|\phi\|_{cb} = inf\{\|\sum_i G_i G_i^*\|^{1/2} \|\sum_i H_i^* H_i\|^{1/2}\},$$

where the infimum is taken over all invertible matrices $S$. It is also enough to consider positive, invertible matrices for $S$.

This algorithm reduces the computation of $\|\phi\|_{cb}$ to a series of matrix computations and only the last step might involve a difficult minimization.

To begin to justify the algorithm, we begin with the last step. First we show that $\phi(X) = \sum_i G_i X H_i$. This can be seen formally, because

$$\sum_i G_i X H_i = (G_1, ..., G_p) \begin{pmatrix} X & 0 & \dots & 0 \\ 0 & X & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & X \end{pmatrix} \begin{pmatrix} H_1 \\ \vdots \\ H_p \end{pmatrix}$$

$$= (E_1, ..., E_p)(t_{i,j} I_n) \begin{pmatrix} X & 0 & \dots & 0 \\ 0 & X & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & X \end{pmatrix} (s_{i,j} I_n) \begin{pmatrix} F_1 \\ \vdots \\ F_p \end{pmatrix}$$

$$= \sum_i E_i X F_i = \phi(X)$$

since the two scalar matrices behave like environmental operators, in the quantum literature sense, and commute past the direct sum in the middle.

Next we need to see that the linear maps from $M_n$ to $M_k, \mathcal{L}(M_n, M_k)$ can be identified with the tensor product, $M_{k,n} \otimes M_{n,k}$ via the map that sends an elementary tensor, $A \otimes B$ to the map $\phi(X) = AXB$. It is easily seen that this extends to a linear map, $\Gamma : M_{k,n} \otimes M_{n,k} \to \mathcal{L}(M_n, M_k)$, that a simple dimension count shows is one-to-one and onto(both spaces have dimension $n^2k^2$).

We now endow $M_{k,n} \otimes M_{n,k}$ with a norm so that $\Gamma$ will be an isometry when $\mathcal{L}(M_n, M_k)$ is endowed with the cb-norm. By the CB representation theorem, we see that if we define for $U \in M_{k,n} \otimes M_{n,k}$,

$$\|U\|_h = \inf\{\|\sum_i A_i A_i^*\|^{1/2} \|\sum_i B_i^* B_i\|^{1/2}\},$$

where the infimum is taken over all ways to represent $U = \sum_i A_i \otimes B_i$ as a sum of elementary tensors, then we will have that $\|U\|_h = \|\Gamma(U)\|_{cb}$.

The above tensor norm is called the **Haagerup tensor norm** in honor of U. Haagerup who was the first to notice the above identification. We write $M_{k,n} \otimes_h M_{n,k}$ to denote the tensor product endowed with this norm and note that we have just proved that:

**Theorem 20** (Haagerup)**.** *The map $\Gamma : M_{k,n} \otimes_h M_{n,k} \to CB(M_n, M_k)$ defined by $\Gamma(A \otimes B)(X) = AXB$ is an isometric isomorphism.*

Here we use $CB(M_n, M_k)$ to denote the space of linear maps from $M_n$ to $M_k$ endowed with the completely bounded norm. The above isomorphism was greatly extended in work of Haagerup and Effros-Kishimoto to other identifications between spaces of completely bounded maps and Haagerup tensor products.

The above theorem reduces the justification of the above algorithm to showing that if $\phi = \Gamma(U)$, then the algorithm correctly computes, $\|U\|_h$. The fact that this algorithm correctly computes $\|U\|_h$ for any operator spaces is proven in [2]. We outline the key ideas below.

For this, we will need a few facts about tensor products of vector spaces.

Recall that if $\mathcal{V}$ and $\mathcal{W}$ are vector spaces, then every element of $\mathcal{V} \otimes \mathcal{W}$ is a finite sum of elementary tensors. The least number of elementary tensors that can be used to represent an element $u \in \mathcal{V} \otimes \mathcal{W}$ is called the **rank of u** and is denoted by **rank(u)**.

**Proposition 21.** [2] *Let $u \in \mathcal{V} \otimes \mathcal{W}$. If $u = \sum_{i=1}^p v_i \otimes w_i$ then $p = rank(u)$ if and only if $\{v_1, ..., v_p\}$ is a linearly independent set and $\{w_1, ..., w_p\}$ is a linearly independent set. Moreover, if $u = \sum_{i=1}^p x_i \otimes y_i$ is another way to represent $u$ as a sum of elementary tensors and $p = rank(u)$, then*

$$span\{v_1, ..., v_p\} = span\{x_1, ..., x_p\}$$

*and*

$$span\{w_1, ..., w_p\} = span\{y_1, ..., y_p\}.$$

**Proposition 22.** [2] *Let $u \in \mathcal{V} \otimes \mathcal{W}$. If we apply Step 1 and Step 2 of the above algorithm to $u = \sum_{i=1}^m a_i \otimes b_i$, to obtain $u = \sum_{i=1}^p e_i \otimes f_i$, then $\{e_1, ..., e_p\}$ and $\{f_1, ..., f_p\}$ will be linearly independent sets and hence $rank(u) = p$.*

These facts are easily proved by applying maps of the form $f \otimes id_W$ and $id_V \otimes g$, where $f$ and $g$ are linear functionals to $u$.

The remainder of the proof of the justification of the algorithm is to show that at each stage, removing the linear dependencies among the elements in the sum for $u$ reduces the Haagerup norm. This is best seen at each stage of the algorithm. Say at Step 1, when we choose the basis, $\{C_1, ..., C_l\}$ and express, $B_i = \sum_j d_{i,j} C_j$, if we first polar decompose the matrix $(d_{i,j}) = (w_{i,j})(p_{i,j})$ where $W = (w_{i,j})$ is an $m \times l$ partial isometry and $P = (p_{i,j})$ is an invertible $l \times l$ positive matrix, then we have that $B_i = \sum_i w_{i,j} \tilde{C_j}$, with $\tilde{C_i} = \sum_j p_{i,j} C_j$. In this case, the set $\{\tilde{C_1}, ..., \tilde{C_l}\}$ is another basis for the span of $\{C_1, ..., C_l\}$ and $\sum_i \tilde{C_i}^* \tilde{C_i} = \sum_i C_i^* C_i$. Moreover, using this basis, we would obtain another representation for $\phi(X) = \sum_{j=1}^{l} \tilde{D_j} X \tilde{C_j}$, where $\tilde{D_j} = \sum_i w_{i,j} A_i$. Again, since $P$ is invertible, the span of $\{\tilde{D_1}, ..., \tilde{D_l}\}$ is the same as the span of $\{D_1, ... D_l\}$. Moreover, since $W$ is a partial isometry, one finds that $\sum_i \tilde{D_i} \tilde{D_i}^* \leq \sum_i A_i A_i^*$. Thus, the infimum of the Haagerup norm expression over all linear combinations of the $D_i$'s and $C_i$,'s which is the same as the infimum over all linear combinations of the $\tilde{D_i}$'s and $\tilde{C_i}$'s is smaller.

This proves that the quantity defining the Haagerup tensor norm(which is the same as the CB norm) must be attained when the coefficients of the generalized Choi-Kraus representation are linearly independent, and hence represented by some choice of basis for $span\{E_1, ..., E_p\}$ and $span\{F_1, ..., F_p\}$.

## 3. The Commutant Representation of a CB Map

In this section we outline an alternative representation to the generalized Choi-Kraus representation of a completely bounded map, that we believe might be of relevance to the quantum information community. The point is that there are two different ways to generalize the Choi-Kraus representation from completely positive mpas to completely bounded maps. This second way might be more relevant from the point of view of environmental operators and has somewhat better uniqueness properties than the representation given in the last section.

It involves a new norm on the space of completely bounded maps, denoted by $|||\phi|||$, that we shall explain later. Let us begin with the representation theorem.

**Theorem 23** (The Commutant Representation). [6] *Let* $\phi : M_n \to M_k$, *then there exists an integer* $m \leq nk$, $A_i \in M_{n,k}, 1 \leq i \leq m$, *and a matrix* $T = (t_{i,j}) \in M_m$ *with* $\sum_i A_i^* A_i = I_k$ *and* $\|T\| = |||\phi|||$, *such that*

$$\phi(X) = \sum_{i,j=1}^{m} t_{i,j} A_i^* X A_j.$$

The name "commutant representation" comes from the fact that

$$\phi(X) = (A_1^*, ..., A_m^*) \begin{pmatrix} t_{1,1} I_n & \dots & t_{1,m} I_n \\ \vdots & & \vdots \\ t_{m,1} I_n & \dots & t_{m,m} I_n \end{pmatrix} \begin{pmatrix} X & 0 & \dots & 0 \\ 0 & X & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & X \end{pmatrix} \begin{pmatrix} A_1 \\ \vdots \\ A_m \end{pmatrix}$$

where the operator $T \otimes I_n$ is an "environment" operator that is in the commutant of the representation of $M_n$, corresponding to the completely positive unital map, $\psi(X) = \sum_i A_i^* X A_i$.

Using duality, one has that $\phi(X) = \sum_{i,j} t_{i,j} A_i X A_j^*$ with $\sum_i A_i^* A_i = I$, precisely when, $|||\phi^\ddagger||| \leq \|T\|$.

There are some analogies with complex measures. The map completely positive map $\psi$ can be thought of as the normalized(so that $\psi(I) = I$) "total variation" of the completely bounded map with the operator $T$ playing the role of the **Radon-Nikodym derivative** $\frac{d\phi}{d\psi}$ in the sense of Arveson[1].

The commutant representation has roughly 1/2 the non-uniqueness of the other generalization of the Choi-Kraus representation, because there is only one completely positive map($\psi$) associated with a completely bounded map $\phi$, albeit still non-unique, as opposed to the two completely positive maps($\phi_1, \phi_2$) needed for the other representation.

A further motivation for this representation comes from analogies with the numerical radius. Recall that for an operator $X \in B(\mathcal{H})$, the **numerical radius** of the operator is defined( in mathematicians notation) by

$$w(X) \equiv \sup\{|\langle Th, h\rangle| : \|h\| = 1\}.$$

The numerical radius is also a norm on $B(\mathcal{H})$ that is equivalent to the operator norm and satisfies,

$$w(X) \leq \|X\| \leq 2w(X),$$

with $w(H) = H$, when $H = H^*$. Thus, both of these norms agree on positive operators and represent two different ways to extend the norm on positive operators to all operators. Consequently, they both have many characterizations in terms of the partial order structure. For example,

$$w(X) = \sup\{\|Re(\lambda X)\| : |\lambda| = 1, \lambda \in \mathbb{C}\} = \inf\{\|P\| : P + Re(\lambda X) \geq 0, \forall |\lambda| = 1\},$$

while,

$$\|X\| = \inf\{\|P\| : \begin{pmatrix} P & X \\ X^* & P \end{pmatrix} \geq 0\}.$$

When one examines these various order theoretic characterizations of the norm for completely bounded maps, with completely positive maps playing the role of the positive operators, one finds that $\|\cdot\|_{cb}$ is the analogue of the numerical radius norm. This lead the authors of [6] to introduce a new norm, the norm $|||\cdot|||$, on the completely bounded maps that is a better analogue of the operator norm.

The key defining property of the triple norm is given below.

**Theorem 24.** [6] *Let $\mathcal{A}$ be a unital $C^*$-algebra and let $\phi : \mathcal{A} \to B(\mathcal{H})$. Then $|||\phi||| \leq 1$ if and only if there exists a unital completely positive map $\psi : \mathcal{A} \to B(\mathcal{H})$, such that the map $\Phi : M_2(\mathcal{A}) \to M_2(B(\mathcal{H}))$ given by $\Phi(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} \psi(a) & \phi(b) \\ \phi^8(c) & \psi(d) \end{pmatrix}$ is completely positive. Moreover, $\|\phi\|_{cb} \leq |||\phi||| \leq 2\|\phi\|_{cb}$.*

The following result is the analogue of the Stinespring representation for the triple norm.

**Theorem 25.** [6] *Let $\mathcal{A}$ be a unital C*-algebra and let $\phi : \mathcal{A} \to B(\mathcal{H})$ be a completely bounded map. Then there exists a Hilbert space $\mathcal{K}$, a unital *-homomorphism, $\pi : \mathcal{A} \to B(\mathcal{K})$, an isometry, $V : \mathcal{H} \to \mathcal{K}$, and an $T$ in the commutant of $\pi(\mathcal{A})$ with $\|T\| = |||\phi|||$, such that $\phi(a) = V^*T\pi(a)V$, for every $a \in \mathcal{A}$.*

## 4. MIN and MAX Operator Spaces and Quantization

Professor Effros presented the general theory of abstract operator spaces in his talk. So I would like to just focus on two ways that one can make operator spaces out of normed spaces and how these two operator spaces measure to a great deal the difference between classical and quantum.

Given any normed space, $X$, if we take an isometric embedding, $\phi : X \to B(\mathcal{H})$, then this allows us to identify $X$ with $\phi(X)$ and consequently induces an operator space structure on $X$. In particular, given $(x_{i,j}) \in M_n(X)$, the induced matrix norm is, $\|(x_{i,j})\|_\phi \equiv \|(\phi(x_{i,j})\|$, where the latter norm is taken as an operator on $B(\mathcal{H} \oplus ... \oplus \mathcal{H})$(n copies).

To see that one such embedding exists, recall the canonical double dual embedding,$\hat{}: X \to X^{**}$. Letting $Y$ denote the unit ball of $X^*$ endowed with the weak*-topology, we have that $Y$ is a compact, Hausdorff space and we can regard,$\hat{}: X \to C(Y)$, as an isometric embedding of the linear space $X$ into the abelian C*-algebra of continuous functions on $Y$. Representing $C(Y)$ on some $B(\mathcal{H})$ yields an isometric embedding of $X$ as operators on a Hilbert space.

The operator space $X$ that one obtains this way is denoted by $MIN(X)$, because of the following properties.

**Proposition 26.** [3] *Let $X$ be a normed space and let $MIN(X) \subseteq C(Y)$ where $Y$ denotes the unit ball of $X^*$ as above.*

- *Given $(x_{i,j}) \in M_n(X)$, one has*

$$\begin{aligned}\|(x_{i,j})\|_{MIN(X)} &= \sup\{\|(f(x_{i,j}))\|_{M_n} : f \in Y\} \\ &= \sup\{\|\sum_{i,j} \lambda_i \mu_j x_{i,j}\|_X : \sum_i |\lambda_i|^2 = \sum_j |\mu_j|^2 = 1\}\end{aligned}$$

- *Given any $\phi : X \to B(\mathcal{H}), \|(x_{i,j})\|_{MIN(X)} \leq \|(\phi(x_{i,j})\|$.*

Once one knows that an isometric embedding of $X$ into some $B(\mathcal{H})$ exists, one can define $MAX(X)$ be setting,

$$\|(x_{i,j})\|_{MAX(X)} = \sup\{\|(\phi(x_{i,j})\| : \phi : X \to B(\mathcal{H})\},$$

where the supremum is taken over all isometric embeddings of $X$ into the operators on some Hilbert space. This is agaian endows $X$ with the structure of an operator space. To see that it is an operator space, one simply considers a large enough direct sum of the maps, $\phi$.

The difference between these two operator space structures in some sense measures the difference between classical and quantum. We illustrate this with an example.

Consider the normed space, $X = \ell_n^1$. Using the canonical basis for $\mathbb{C}^n$, we have that $\|\sum_{k=1}^n \lambda_k e_k\|_1 = \sum_{k=1}^n |\lambda_k| = \sup\{|\sum_{k=1}^n \lambda_k e^{i\theta_k}| : \theta_k \in \mathbb{R}\}$. If we let $\mathbb{T}$ denote the unit circle in the complex plane, so that $\mathbb{T}^n$ is the n-torus, and let $z_1, ..., z_n$ denote the coordinate functions on $\mathbb{T}^n$, then the map $\phi(\sum_k \lambda_k e_k) = \sum_k \lambda_k z_k$ can be seen to be an isometric embedding of $\ell_n^1$ into $C(\mathbb{T}^n)$. In fact, the image of $\ell_n^1$ under this embedding can be identified with $MIN(\ell_n^1)$. If we represent $(x_{i,j}) \in M_m(\ell_n^1) = M_m \otimes \ell_n^1$ as $(x_{i,j}) = \sum_{k=1}^n A_k \otimes e_k$ with $A_k \in M_m$, then it follows that,

$$\|\sum_k A_k \otimes e_k\|_{MIN(\ell_n^1)} = \|\sum_k A_k z_k\|_{M_m(C(\mathbb{T}^n))}$$
$$= \sup\{\|\sum_k A_k e^{i\theta_k}\|_{M_m} : \theta_k \in \mathbb{R}\}.$$

Thus, the space $MIN(\ell_n^1)$ is "universal" for n commuting unitaries. Perhaps, not too surprisingly, it can be shown that [4], for $\sum_k A_k \otimes e_k \in M_m(\ell_n^1)$, we have that,

$$\|\sum_k A_k \otimes e_k\|_{MAX(\ell_n^1)}$$
$$= \sup\{\|\sum_k A_k \otimes U_k\|_{M_m(B(\mathcal{H}))} : U_k \in B(\mathcal{H}), U_k^* U_k = U_k U_k^* = I\},$$

i.e., the "universal" norm for n arbitrary unitaries.

For any normed space $X$, we have the constant[4],

$$\alpha(X) = \sup\{\frac{\|(x_{i,j})\|_{MAX(X)}}{\|(x_{i,j})\|_{MIN(X)}} : \forall (x_{i,j}) \in M_m(X), \forall m\},$$

which in some sense measures how much norms change when quantum variables replace classical variables. For our example, it is known that $\sqrt{n/2} \leq \alpha(\ell_n^1) \leq \sqrt{n}$, so that when n commuting unitaries are replaced in formulas by n arbitrary unitaries, the norms will change by at most a factor of $\sqrt{n}$.

There is a characterization of the norm on $MAX(X)$ in [5] that is often useful. For $(x_{i,j}) \in M_n(X)$, we have that $\|(x_{i,j})\|_{MAX(X)} < 1$ if and only if there exists an integer $p$, scalar matrices, $(\lambda_{i,j}) \in M_{n,p}, (\mu_{i,j}) \in M_{p,n}$ with $(\lambda_{i,j})\|\|(\mu_{i,j})\| \leq 1$ and vectors $y_1, ..., y_p \in X$ with $\|y_i\| < 1$, such that $x_{i,j} = \sum_{k=1}^p \lambda_{i,k} y_k \mu_{k,j}, \forall i, j$.

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