Depth of Dead Ends in Cayley Graphs

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Notations and Conventions

Notations

- Γ is always a graph.
- $V(\Gamma)$ is its set of vertices, $E(\Gamma)$ is its set of edges.
- $d(\cdot, \cdot) = d_\Gamma(\cdot, \cdot)$ denotes the induced metric on $V(\Gamma)$.
- $B_n(v) := \{w \in V(\Gamma) | d(v, w) \leq n\}$.

Let G be a group and X be a finite set of generators. The Cayley graph $\Gamma(G, X)$ is defined by $V(\Gamma(G, X)) := G$ and $E(\Gamma(G, X)) := \{(g, gx) | g \in G, x \in X\}$.

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Let Γ be a graph and $v_0 \in V(\Gamma)$. A vertex $v_1 \in V(\Gamma)$ is a *dead end* with respect to v_0 if $B_1(v_1) \subseteq B_{d(v_0,v_1)}(v_0)$.

If Γ is a Cayley graph $\Gamma(G, X)$, we always choose $v_0 = 1_G$

Examples

- $\Gamma(\mathbb{Z}, \{2, 3\})$
  $d(0, 1) = 2$ and $d(0, 1 + 2) = d(0, 1 - 3) = 1$, $d(0, 1 - 2) = d(0, 1 + 3) = 2$. Hence 1 is a dead end.
- $\Gamma(\mathbb{Z}, \{3, 5\})$
  $d(0, 4) = 4$. Adding ± 3 or ± 5 reduces the distance, because $4 = 2 \cdot 5 - 2 \cdot 3 = 3 \cdot 3 - 1 \cdot 5$.

The second example is worse. Starting in 4 one needs even 3 steps to leave $B_4(0)$.

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Let v_1 be a dead end with respect to v_0. The *depth* of v_1 is the maximal nonnegative integer k such that $B_k(v_1) \subseteq B_{d(v_0,v_1)}(v_0)$.

Remarks

- In the upper examples the depth of 1 in $\Gamma(\mathbb{Z}, \{2,3\})$ is 1, and the depth of 4 in $\Gamma(\mathbb{Z}, \{3,5\})$ is 2.
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First statement

Theorem

Let Γ be a graph with more than one end. Then the depth of dead ends in Γ is bounded.

Proof of Theorem

Sketch of proof

Choose $k > 0$ such that for one (hence for each) $v_1 \in V(\Gamma)$ the graph $\Gamma \setminus B_k(v_1)$ has more than one unbounded component. We calculate the depth of v_1 with respect to v_0.

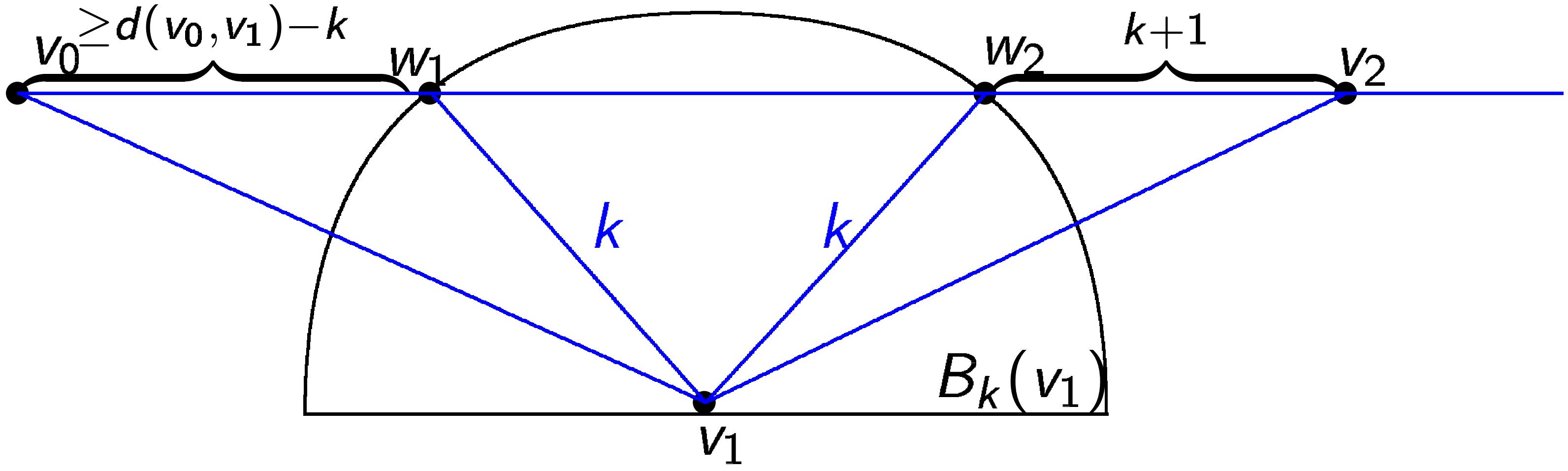

Figure: The depth of the dead end v_1 is bounded above by $2k$: The ray starting in v_0 and ending in a different component of $\Gamma \setminus B_k(v_1)$ has to hit the ball $B_k(v_1)$. The distance $d(v_0, v_2) \geq d(v_0, w_1) + d(w_2, v_2) \geq d(v_0, v_1) + 1$ and $d(v_1, v_2) \leq 2k + 1$.

Bad news

Known results

- Existence of dead ends is not a property preserved by quasi-isometries.
- Results concerning dead ends in Cayley graphs were obtained (a.o.) by Bogopolski (97), Cleary Taback (04), Cleary Riley (04&05), L. (07), Sunic (07), Warshall (07).
- Even the property of having dead ends of unbounded depth (in a Cayley graph) is not a group invariant. [Riley Warshall(06)]
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A further example

Definition (Quasi-Automorphism)

Let Γ be a graph. A quasi-automorphism ϕ of Γ is a bijection of $V(\Gamma)$ which respects all but finitely many edges, e.g.
$|\{(v_1, v_2) \in E(\Gamma) | (\phi(v_1), \phi(v_2)) \notin E(\Gamma)\}| < \infty$ and
$|\{(v_1, v_2) \notin E(\Gamma) | (\phi(v_1), \phi(v_2)) \in E(\Gamma)\}| < \infty$.
The set of all quasi-automorphisms of Γ forms a group called $QA(\Gamma)$.

The example

We now study $G = QA(\Gamma(\mathbb{Z}, \{1\}))$.

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Some facts

- G is generated by $s, t : \mathbb{Z} \to \mathbb{Z}$

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\begin{aligned}
s(k) &:= k+1 \\
t(k) &:= \begin{cases} k & k \neq 0,1 \\ 0 & k = 1 \\ 1 & k = 0 \end{cases}
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- There exists a nice description of $\Gamma(G, \{s, t\})$
  - Vertices: A vertex is a bi-infinite list of all nonzero integers, almost all in increasing order, marked with an arrow, e.g. $(\ldots, -5, -4, 3, 6, -3, -2, -1 \overset{\downarrow}{} 4, 2, 1, 5, 6, \ldots)$
  - Edges: Traveling along an s-edge moves the arrow one step to the left. Traveling along a t-edge swaps the numbers next to the arrow

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In this description the identity is represented by
$v_0 = (\ldots - 3, -2, -1^{\downarrow} 1, 2, 3 \ldots)$.

Theorem

The vertex
$v_k = (\ldots, -(k+1), k, k-1, \ldots 1^{\downarrow} - 1, -2, \ldots - k, k+1, \ldots)$
is a dead end of depth at least $2k$ with respect to v_0.
Furthermore, any path from v_k to a point of $\Gamma \setminus B_{d(v_0, v_k)}(v_0)$ has to enter the ball $B_{d(v_0, v_k)-k}(v_0)$.

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Furthermore, any path from v_k *to a point of* $\Gamma \setminus B_{d(v_0,v_k)}(v_0)$ *has to enter the ball* $B_{d(v_0,v_k)-k}(v_0)$.

Strong depth

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Let $v_0, v_1 \in V(\Gamma)$, and $n = d(v_0, v_1)$. The strong depth of v_1 with respect to v_0 is defined as the minimal number k such that v_1 can be connected to a point of $\Gamma \setminus B_1(n)$ inside $\Gamma \setminus B_1(n-k)$.

Open question

Is the property of having dead ends of arbitrary high strong depth invariant under quasi-isometries?

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The End

Thank you for you attention.