Locally Finite Graphs with Ends – a survey

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Part 1. Concepts

Graph $G \to$ topological space $|G|$ $\quad (= G + \text{ends})$

paths in $G \quad \to \quad$ arcs in $|G|$

cycles in $G \quad \to \quad$ circles in $|G|$

spanning trees in $G \quad \to \quad$ TSTs in $|G|$

+ related concepts

Part 2. Applications & techniques – and open problems

Part 3. The topological viewpoint

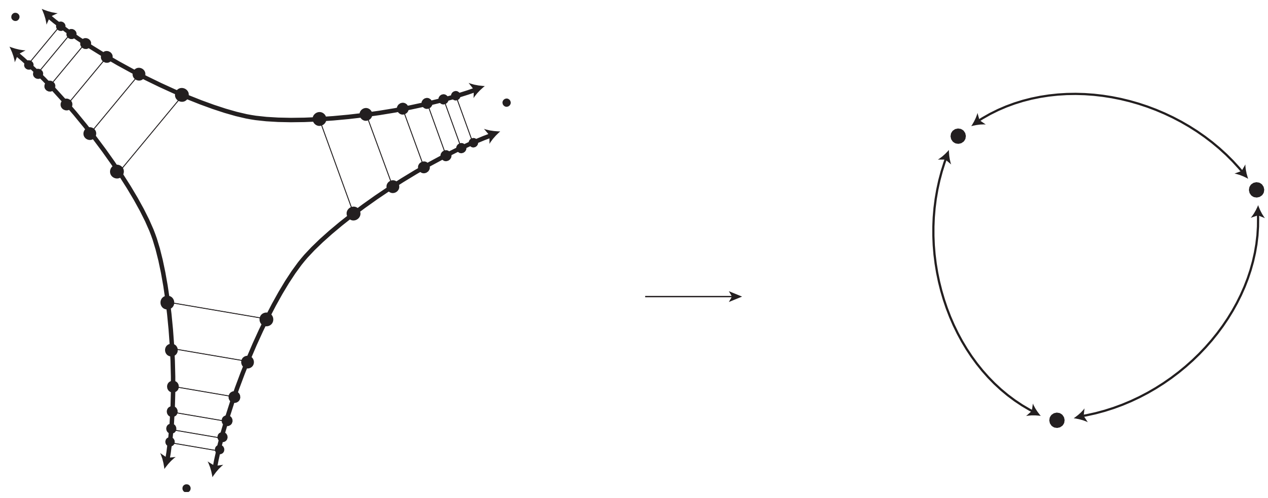
– standard homologies for G and $|G|$

– a new homology to capture $\mathcal{C}(G)$

(Can topology help with cycles, Euler tours, flows, χ-φ duality. . . ?)

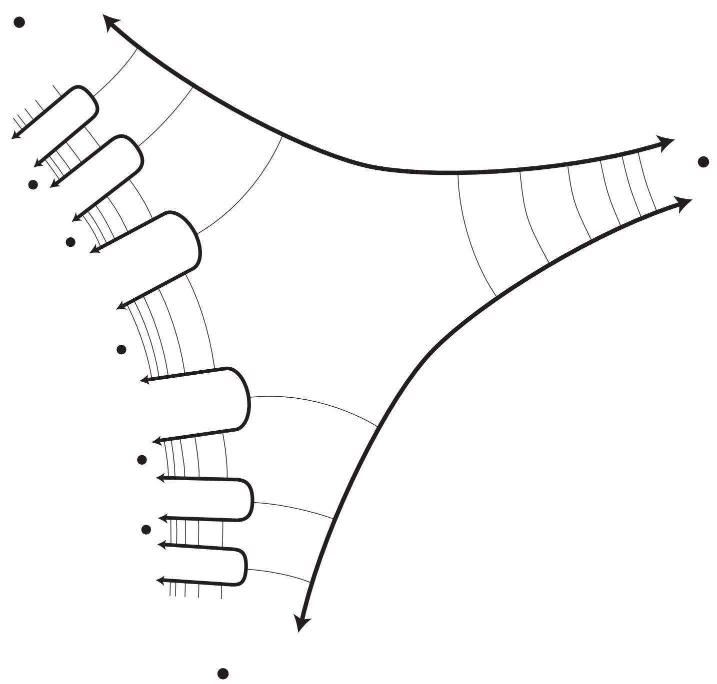
Arcs and circles, naively

Initial idea:



A 'Hamilton circle' through 3 ends

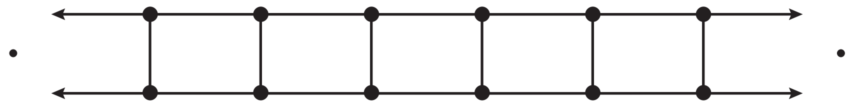
Iterated idea:



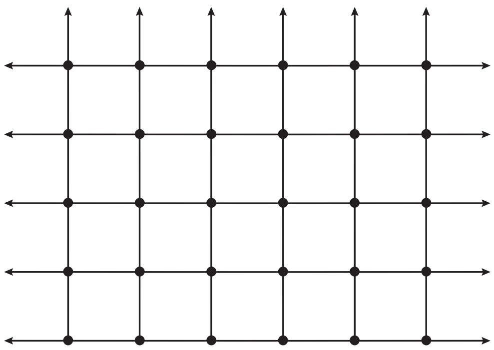
$\Rightarrow$ no idea

The Freudenthal compactification $|G|$ of G

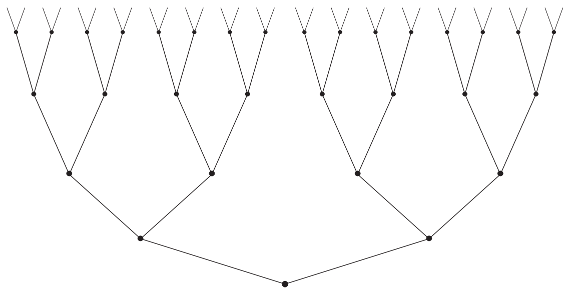
The *ends* of G are its equivalence classes of rays (1-way ∞ paths), where $R \sim R'$ iff no finite set of vertices separates R from R'.



2 ends



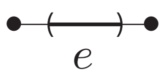
1 end

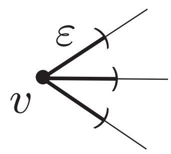


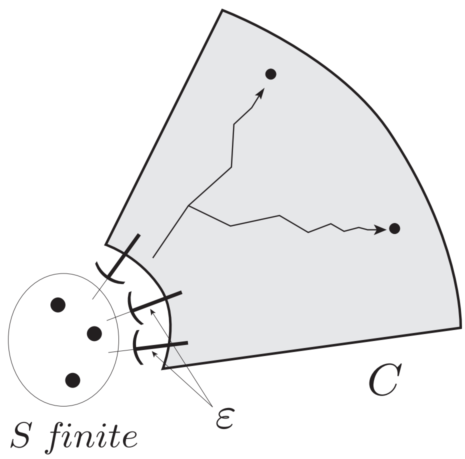
$2^{\aleph_0}$ ends

Points of $|G|$: G as a 1-complex, + ends

Basic open sets:







$\Rightarrow$ every ray converges to 'its' end

Lemma. $|G|$ *is compact.* (For G locally finite and connected)

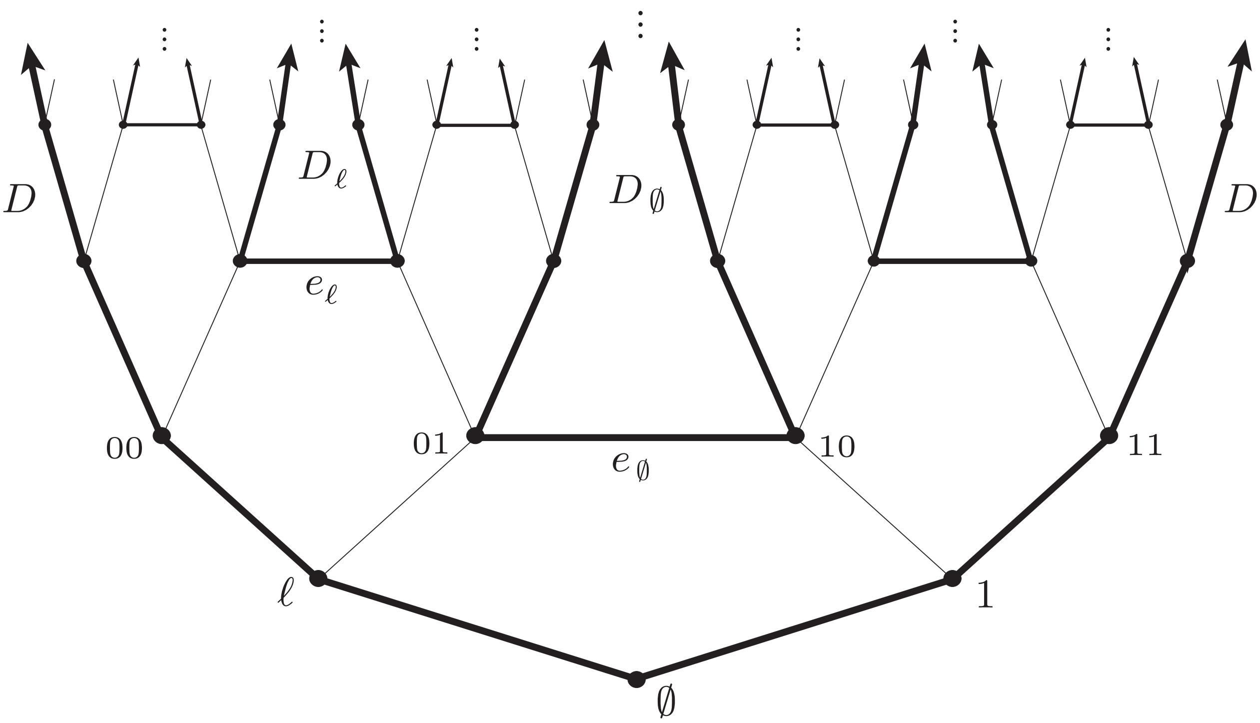
Arcs and circles, topologically

Arc: 1–1 cont's image in $|G|$ of $[0,1]$

Circle: 1–1 cont's image in $|G|$ of S^1

$\Rightarrow$ all our naive 'circles' *are* circles.

Any other arcs or circles?



The 'wild circle' W

Such 'cycles' are necessary!

Jumping Arc Lemma. *Let $\{U, W\}$ be a bipartition of $V(G)$ into connected sets. Iff the U–W cut F consists of finitely many edges, every U–W arc in $|G|$ contains an edge from F.*

Combinatorial degree of an end ω:

vertex-degree: max # disjoint rays in ω

edge-degree: max # edge-disjoint rays in ω

Topological degree of an end ω:

vertex-degree: max # disjoint arcs in ω

edge-degree: max # edge-disjoint arcs in ω

Topological degrees make sense in subgraphs $H \subseteq G$:

– consider arcs in $\overline{H} \subseteq |G|$, *but always the ends of* G.

Example:

$\overline{H}$ is a circle $\Leftrightarrow$ $\overline{H}$ is topologically connected and every vx and end in $\overline{H}$ has (top) degree 2

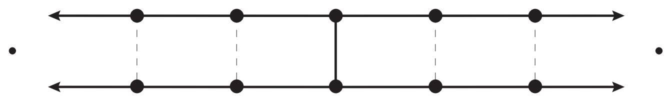
For $H = G$ (loc.finite), comb/top end degrees coincide.

$\Rightarrow$ only topological degrees are needed

TSTs: topological spanning trees

Definition. *A TST is an arc-connected standard subspace of $|G|$ containing all vertices and ends but no circle.*

NB: Standard subspaces containing all ends are closed.
For closed subspaces: connected $\Rightarrow$ arc-connected.

Not a TST: 

Two TSTs:

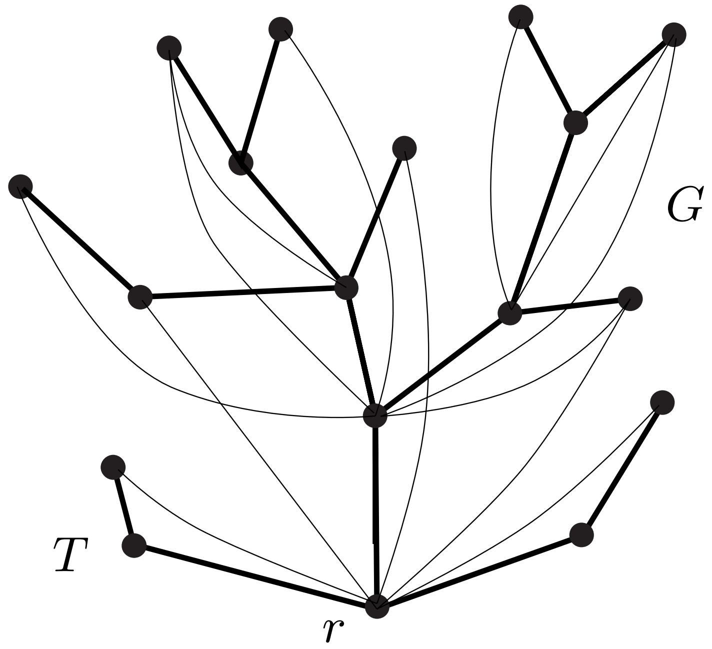
Theorem. *For closed standard subspaces $T \subseteq |G|$ containing all the vertices, the following are equivalent:*

- *T is a TST;*
- *T is edge-maximal without a circle;*
- *T is edge-minimally arc-connected;*
- *Any two points of T are joined by a unique arc in T.*

Fundamental cuts of TSTs are finite.

For spanning trees T: $\overline{T}$ *is a TST* $\Leftrightarrow$ T *is 'end-faithful'.*

In particular, *normal* spanning trees (NSTs) have TST closures.

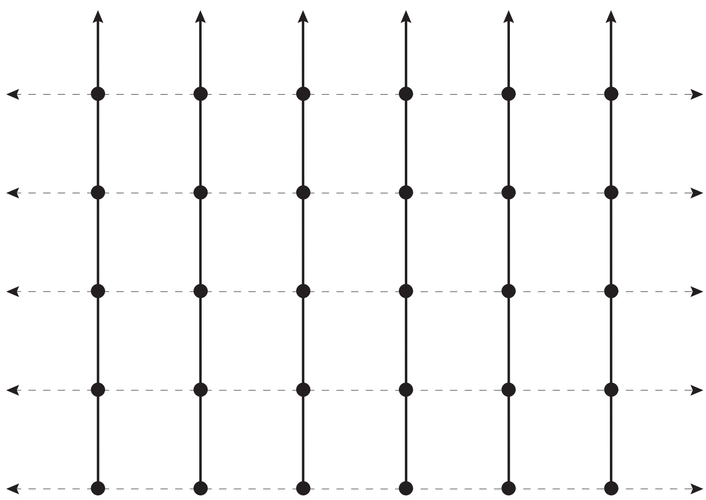


NSTs always exist, and are the most useful TSTs:

T an NST $\Rightarrow$ $|G|$ has the 'same' basic open sets as $|T|$

(their vx sets are 'up-trees' $\lfloor t \rfloor$, for $S := \{s \mid s < t\}$)

But there are other TSTs:



A 'disconnected' TST

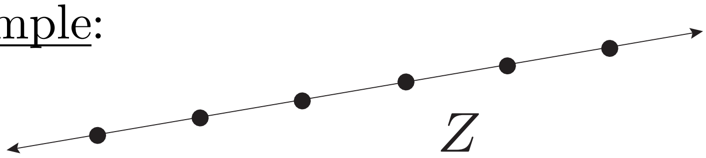
The topological cycle space

$$\mathcal{C}_{\text{fin}}(G) := \langle\, E(C) \mid C \text{ cycle in } G \,\rangle_{\text{finite sums mod 2}}$$

$$\mathcal{C}_{\text{top}} := \langle\, E(C) \mid C \text{ circle in } |G| \,\rangle_{\text{thin infinite sums mod 2}}$$

circuit : edge-set of circle

<u>Example</u>:



$E(Z) \notin \mathcal{C}$ $(:= \mathcal{C}_{\text{top}})$

Properties of $\boldsymbol{\mathcal{C}}$ $(:= \mathcal{C}_{\text{top}})$

- The fundamental circuits of any TST generate $\mathcal{C}$
  Proof: $\mathcal{C} \ni C = \sum_{e \in C \smallsetminus T} C_e$ (needs jump.arc & fund.cuts finite)

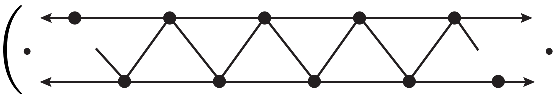
1. Works *only* for TST; 2. Cor for NST: $\mathcal{C}$ generated by finite circuits

- $\mathcal{C} = \{\,\text{finite cuts}\,\}^{\perp}$ and $\mathcal{C}_{\text{fin}} = \{\,\text{cuts}\,\}^{\perp}$

  $\{\,\text{finite cuts}\,\} = \mathcal{C}^{\perp}$ and $\{\,\text{cuts}\,\} = \mathcal{C}_{\text{fin}}^{\perp}$

? • $\mathcal{C} = \{\, F \subseteq E(G) \mid d_{(V,F)}(x) \text{ is even } \forall\, x \in V \cup \Omega \,\}$

  – even vx degrees not enough: 

  – end degrees are *edge*-degrees: $E\Big($  $\Big) \notin \mathcal{C}$

  – even/odd defined even for infinite degrees of ends

  – known only for $F = E(G)$ MAJOR OPEN PROBLEM!

- Every $D \in \mathcal{C}$ is a *disjoint* union of circuits.

Compactification vs. metric completion

For G locally finite, $|G|$ is metrizable. Generally:

Theorem. $|G|$ *metrizable* $\Leftrightarrow$ G *has an NST.*

'$\Leftarrow$': NST $\rightarrow$ for $e \in E(T)$ let $\ell(e) := 2^{-\text{height}(e)}$

$\rightarrow$ for $x, y \in V(G) \cup \Omega(G)$ let $d_\ell(x, y) := \sum_{e \in xTy} \ell(e)$

$\rightarrow$ metric on $|G|$ inducing the correct topology

$|G|$ compact $\Rightarrow$ complete as a metric space

$\Rightarrow$ $|G|$ is the (unique) completion of the metric space (G, d_ℓ)

Trivially, the above d_ℓ also satisfies $\forall\, u, v \in V(G)$:

$$d_\ell(u, v) = \inf \sum_{e \in P} \ell(e) \text{ over all } u\text{–}v \text{ paths } P \text{ in } G. \qquad (*)$$

Conversely, given *any* function of edge lengths $\ell \colon E(G) \rightarrow (0, 1]$, $(*)$ defines a metric d_ℓ on G, and we can study its completion.

Theorem. *Whenever* $\ell \colon E(G) \rightarrow (0, 1]$ *satisfies* $\sum_{e \in G} \ell(e) < \infty$, *the completion of* (G, d_ℓ) *coincides with* $|G|$.

How about other metrics on G?

$\rightarrow$ go to Angelos' workshop...

Graphs with Ends II: applications and techniques

1. Cycle space applications

A *topological Euler tour through* $F \subseteq E(G)$ is a closed topological path in $|G|$ that is injective inside edges, traverses every edge in F exactly once, and traverses no other edge.

Traditional: ask for Eulerian double rays. Fails if G has $\geqslant 3$ ends.

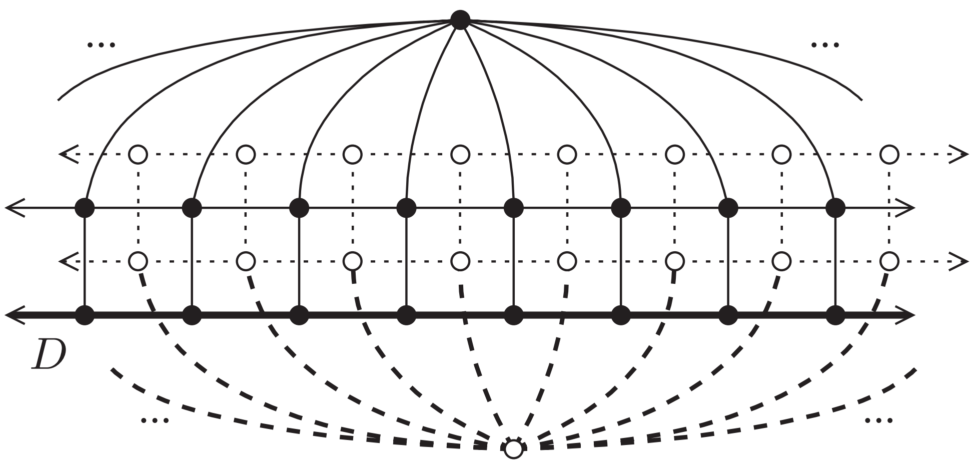
'Euler's theorem'.

$|G|$ contains a topological Euler tour through F iff $F \in \mathcal{C}(G)$.

Not clear why this should be true: we can't just 'concatenate' ∞'ly many circuits, eg disjoint ones, blindly: the topology has to be 'right'.

Call G^* a *dual* of G if $E(G^*) = E(G)$ and the bonds (min'l cuts) of G^* are precisely the circuits of G. These may be infinite. We have to allow certain non-locally finite graphs, and adjust $|G|$.

($*$) No two v's are joined by ∞'ly many edge-disjoint paths. USE IToP!



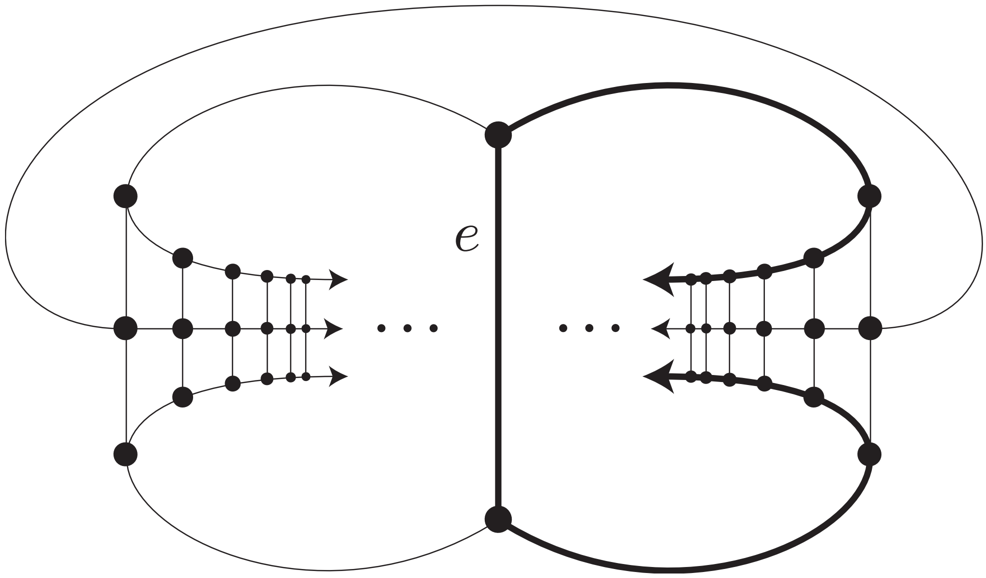
'Whitney's theorem'.

G has a dual iff G is planar. When G is 3-connected, this dual G^* is unique (and 3-connected), and $G^{**} = G$.

Uniqueness and $G^{**} = G$ fail for duality of only finite cuts/circuits.

A family $\mathcal{F}$ of edges sets is *sparse* if no edge lies in > 2 elt's of $\mathcal{F}$.
Example: facial circuits in (finite) plane graphs.

For $\mathcal{C}_{\rm fin}$, '$\Rightarrow$' of ML fails below: we need ∞ face bdries, even to generate $\mathcal{C}_{\rm fin}$ sparsely.



'MacLane's theorem'.

G is planar iff $\mathcal{C}(G)$ has a sparse generating subset.

A cycle/circle C is *peripheral* if C has no chord and $V(C)$ does not separate G.

For $\mathcal{C}_{\rm fin}$, '$\Leftarrow$' of K–T fails: too many ∞ periph'l circles can kill planarity too.

'Kelmans–Tutte theorem'.

G (3-conn'd) is planar iff every edge lies on $\leqslant 2$ peripheral circles.

'Tutte structure theorem'.

G 3-connected $\Rightarrow$ the peripheral circuits generate $\mathcal{C}(G)$.

The ML fig above shows both that ∞ circuits are needed to generate (as e lies in no finite peripheral circuit), and that ∞ sums are needed (to generate any finite circuit through e).

'Gallai's partition theorem'.

$E(G)$ either lies in $\mathcal{C}(G)$ or partitions into a cut and two elements of $\mathcal{C}(G)$ each induced by one side of the cut.

2. Applications in 'extremal' infinite graph theory

Forcing local structure (K_n minor) by global assumptions ('many edges'), or forcing global structure (Hamilton cycle) by local assumptions (min.deg)

Two reasons why there is no infinite extremal graph theory:

- need 'more paths and cycles' (as in ML etc)
- 'many edges', large δ $\quad\not\Rightarrow\quad$ anything (eg, dense minors)

draw tree of large min.deg here

$\Rightarrow$ need high-degree ends as 'wrapper'

Theorem.

(i) *If $\delta(G) \geqslant 2k^2 + 6k$ and every end of G has vertex-degree at least $2k^2 + 2k + 1$, then G has a $(k+1)$-connected subgraph.*

(ii) *If $\delta(G) \geqslant 2k$ and every end of G has edge-degree $\geqslant 2k$, then G has a $(k+1)$-edge-connected subgraph.*

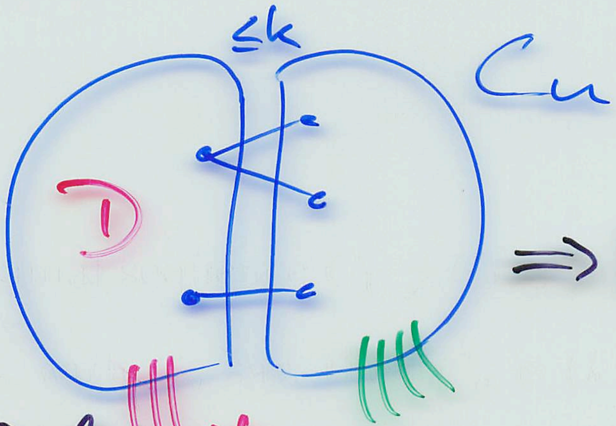
($\bigcup\!\!\!*$ & end degrees $\geq 2k$ $\Rightarrow \exists$ $(k+1)$-edge-conn^d subgr.)

Proof of (ii):

Consider a maximal sequence $C_1 \overset{\text{conn}^d}{\supsetneq} C_2 \overset{\text{conn}^d}{\supsetneq} \ldots$ with $|\partial C_n| < 2k \;\forall n$.

If it terminates, with C_n say, then C_n is $(k+1)$-edge-connected:

else:

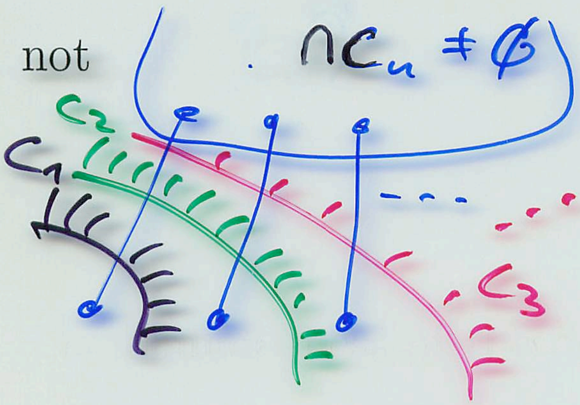


$\Rightarrow |\partial D| < 2k$ ↯

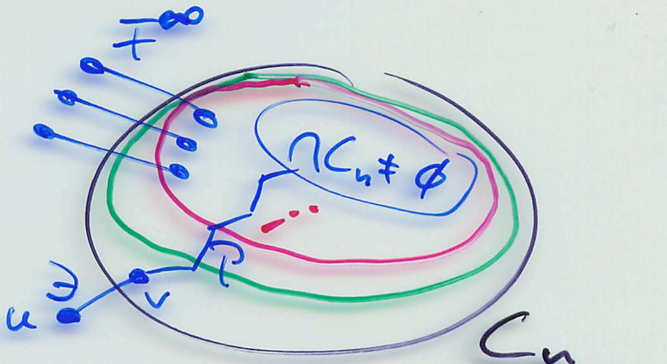
$|\underbrace{\partial C_n}_{\text{|||} + \text{||||}}| < 2k \Rightarrow$ wlog $\text{|||} < k$

The boundedness of the ∂C_n implies that $\bigcap C_n = \emptyset$:

not



& not

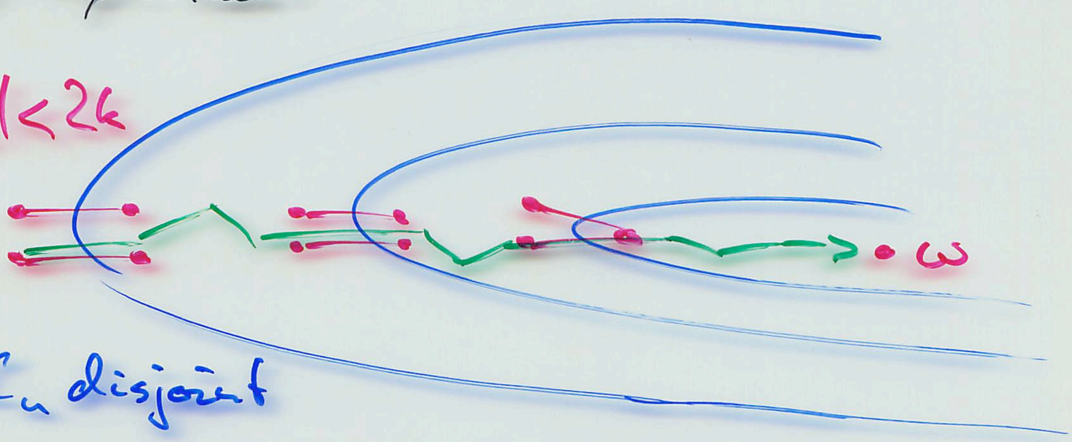


n large enough that $F^\infty \subseteq \partial C_n$

$\Rightarrow P$ meets ∞'ly many disj't $\partial C_n \setminus F^\infty$ ↯

but $F^\infty = \emptyset$. Then:

$|\partial C_n| < 2k$



all ∂C_n disjoint

$\Rightarrow \;\; \exists$ an end whose edge-degree is $< 2k$.

Example: Tree-Packing

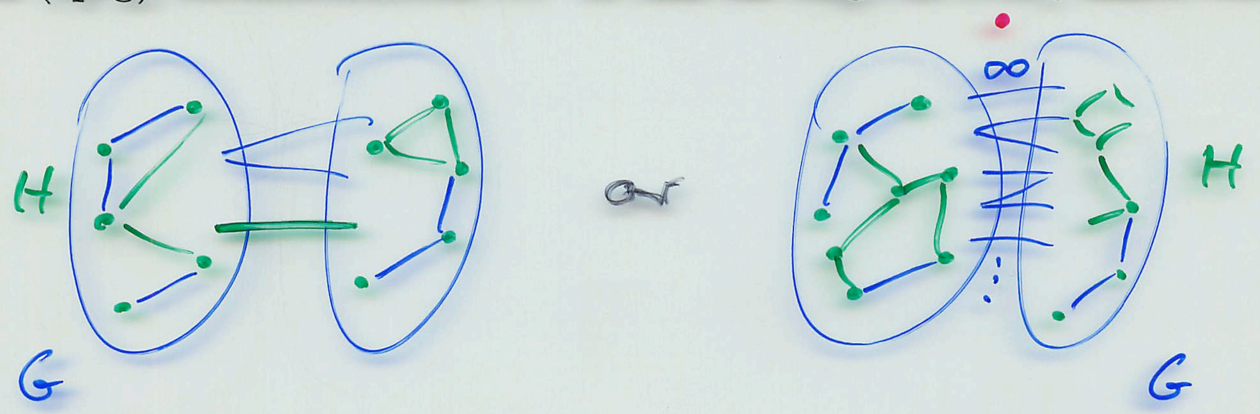
Theorem 1. (Nash-Williams 1961; Tutte 1961)
The following are equivalent for a finite multigraph G and $k \in \mathbb{N}$:

- *G has k edge-disjoint spanning trees.*
- *For every vertex partition, into ℓ sets say, G has at least $k(\ell-1)$ edges between different partition sets.*

Theorem 2. (Tutte 1961)
The following are equivalent for all locally finite G and $k \in \mathbb{N}$:

- *G has k edge-disjoint spanning* <u>*semiconnected*</u>** subgraphs.*
- *For every vertex partition, into ℓ sets say, G has at least $k(\ell-1)$ edges between different partition sets.*

*) H (sp'g) *semiconnected* $:\Leftrightarrow$ H has an edge in every finite cut of G



$\Leftrightarrow$ the closure $\overline{H}$ of H in $|G|$ is (topologically) connected!

$\Leftrightarrow$ $\overline{H}$ contains a TST.

Theorem 2′.
The following are equivalent for all locally finite G and $k \in \mathbb{N}$:

- *G has k edge-disjoint TSTs.*
- *For every vertex partition, into ℓ sets say, G has at least $k(\ell-1)$ edges between different partition sets.*

<u>Cor</u>: G $2k$-edge-conn^{d} $\Rightarrow$ $\exists$ k edge-disj^{t} TSTs

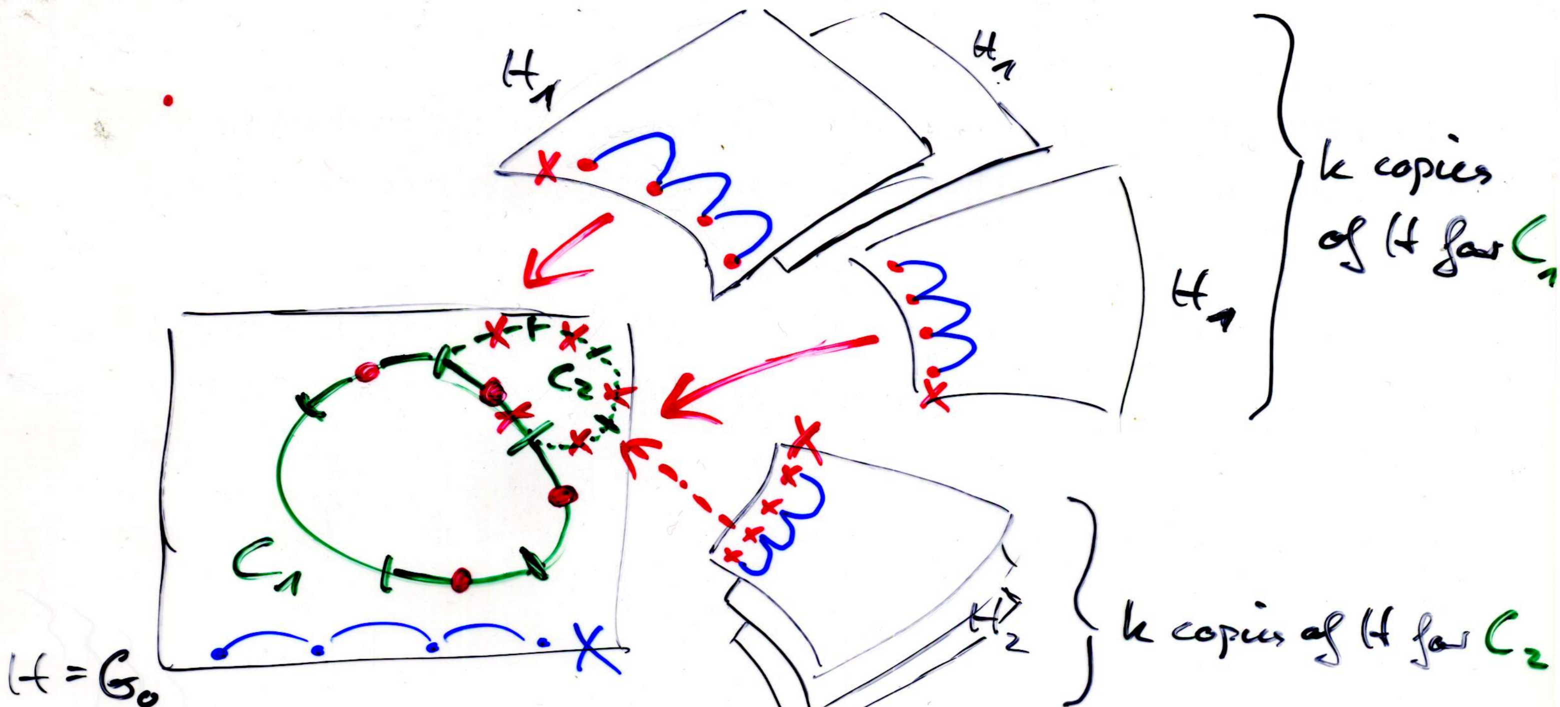
The Aharoni-Thomassen Construction (1989)

A locally finite k-connected graph G without non-separating cycles:

1. Start with a copy G_0 of a k-connected graph H of girth $\geqslant k^2$. Let X be a set of k vertices in H at pairwise distance $\geqslant k$.

2. From each cycle in G_0 pick k edges, subdivide them. Identify the k new vertices with X in each of $\geqslant k$ new copies of H.

3. Repeat ω times, grafting new copies of H only on to edges added at the previous step (for each cycle in current graph).



G cannot have > 2 edge-disjoint spanning trees, because the edges of any fundamental cycle separate G but come from only 2 trees.

- $\Rightarrow$ *Need TSTs* ('more paths') to make Thm 2′ true.
- $\Rightarrow$ *Need circles* ('more cycles') to assume the role played by non-separating cycles in finite graphs. (G has non-sep'g circles!)

Georgakopoulos (2003): ↑ can be done in the plane

Hamilton circles

Conjecture. *G planar, 4-connected $\Rightarrow$ $|G| \supseteq$ Hamilton circle.*

Progress: Yu's talk

Conjecture. *G 2-connected $\Rightarrow$ $|G^2|$ has a Hamilton circle.*

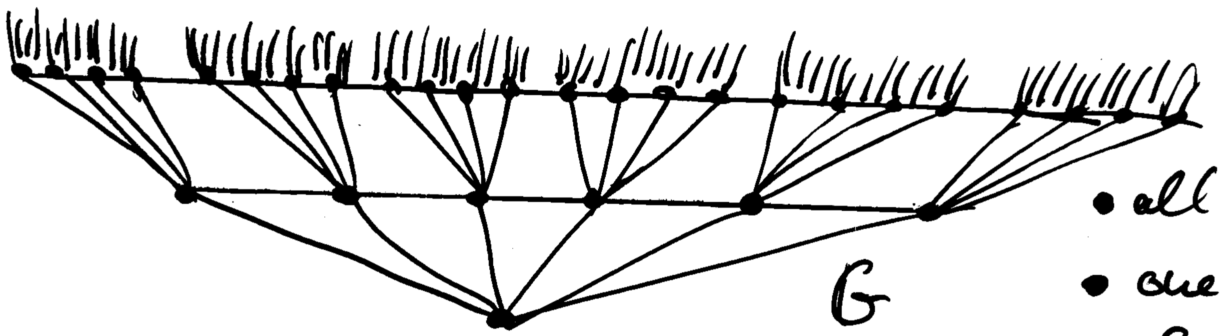
Proof: Georgakopoulos' talk

Problems. *Let G be countable but not necessarily locally finite.*

- *G is connected $\Rightarrow$ $|G^3|$ has a Hamilton circle.*
- *G is 2-connected $\Rightarrow$ $|G^2|$ has a Hamilton circle.*
- *If $|G^d|$ has a Hamilton circle then so does $|G^{d+1}|$.*

Basic "extremal" questions:

Large vertex- and end-degrees cannot force $H \subseteq G$ (or $H \xhookrightarrow[\text{top}]{} |G|$) for non-planar H:



- all $d(v)$ large
- one end only, of ∞ degree

- Which planar H can we force in $|G|$ by degree assumptions on G?

- When is $H \xhookrightarrow[\text{top}]{} |G|$ easier to force than $TH \subseteq G$?

- Find conditions (on G, ends of G, $|G|, \dots$) that do force, e.g., $K_n \xhookrightarrow[\text{top}]{} |G|$!

3. Techniques

Constructing arcs and TSTs greedily

…usually fails.

Example 1: constructing TSTs from below

(Prove: *Every acirclic standard subspace of* $|G|$ *extends to a TST.*)
Circles arise at limit step if finite T_n are chosen greedily = 'blindly'.

Example 2: constructing an arc by extension

(Prove: *Every* $D \in \mathcal{C}$ *contains a circle through any given* $e \in D$*.*)
Finite: just extend e to path in $\bigcup D$ until closed. ∞: can't find wild circle like this – how emerge from an end? BUT: the wild circle is the union of finite face bdries, hence in $\mathcal{C}$, so it MUST contain a circle through every edge!

The use of compactness

Challenge: additional requirements on the limit
such as 'continuity at ends'. A typical assertion desired for the limit is not 'finitary', ie can fail even if all its finite restrictions are true.
$\Rightarrow$ constructions by compactness, not proofs

Limits of edge sets

Example 3: TSTs from above

(Thm: *Given standard subspaces* $X \subseteq Y \subseteq |G|$ *with* X *acirclic and* Y *spanning (v's and ends) and connected, there is a TST,* T *say, such that* $X \subseteq T \subseteq Y$*.*)

General technique:

Approximate G by G_n ($n = 1, 2, \dots$): contract components of $G - G[v_1, \dots, v_n]$, keeping parallel edges but deleting loops:

figure of G_n

Example 4 (simple compactness): trying to contruct a circuit, but finding just a set $D \in \mathcal{C}$.

Given $\forall\, n$: a circuit $C_n \in \mathcal{C}(G_n)$

Note: for $m < n$, cut criterion $\Rightarrow$ $C_n \cap E(G_m) \in \mathcal{C}(G_m)$

(The cuts of G_m are also cuts of G_n, so C_n meets them evenly.)

Compactness yields nested $D_n \in \mathcal{C}(G_n)$ with $D := \bigcup_n D_n \in \mathcal{C}(G)$.

Again by cut criterion: every finite cut of G is also a cut of every G_n with n large enough, and as the D_n for those n meet it evenly so does D.

Example 5: really constructing a circle, or u–v arc in $X \subseteq |G|$,

Given $\forall\, n$: some u–v path $P_n \subseteq X \cap G_n$ (= cycle through uv)

Note: for $m < n$, P_n induces an u–v walk on G_m

$\rightarrow$ what can we say about a limit of such walks?

1st answer: its closure X is top. connected (edge-Menger),

$\underset{\text{lemma}}{\Rightarrow}$ arc-connected $\Rightarrow$ $\exists$ u–v arc.

2nd answer: below

Limits of paths

Idea: in our sequence of walks W_n , not only $E(W_n) \subseteq E(W_{n+1})$ but $W_n \to W_{n+1}$ by expanding a dummy vertex of G_n.

→ parametrize W_n as top.path, obtain limit path (continuous?)

→ extract u–v arc (lemma).

Example 6 (simpler, but same principle): tour around T_2

Proof for Example 6: The task is to define a closed top'l path that traverses every edge exactly twice. To define this in a limit process, walk around a finite subtree in this manner, pausing at every leaf for a non-trivial time interval. At the next step, expand that interval to a walk around the up-tree of height 1 at that vx, again pausing at every leaf. For some $x \in [0, 1]$, the image gets redefined infinitely often. But then these images map out an upward ray in T_2, and we let the limit map map x to the end ω of that ray. Then prove that the limit map is cont's at such x. (It clearly is elswhere.) The proof is nearly the same: given a nbhd $\hat{C}$ of ω, take an interval around x in $[0, 1]$ small enough that some σ_n maps it to a vx in C. Then every σ_m with $m > n$ maps x to some point in the up-tree of that vertex (possibly an end), and hence also to $\hat{C}$.

III The topological viewpoint

Today:

- Reminder of singular homology (basic def^s only)

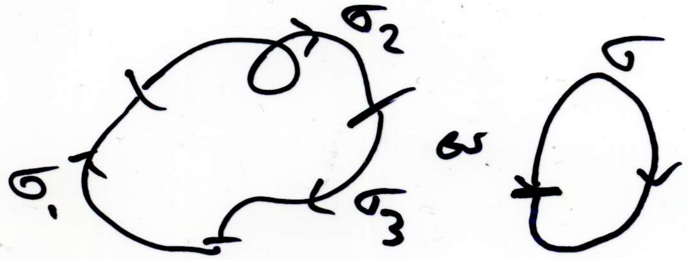
- Singular homology of $|G|$:

$$H_1(|G|) \overset{?}{\longleftrightarrow} \mathcal{C}_{top}(G)$$

- A new singular homology for locally finite CW-complexes capturing precisely $\mathcal{C}$

5"

$H_1(\lvert G\rvert)$	$\mathcal{C}_{top}(G)$
Similarities:	S^1
gen'd by elementary 1-cycles: $\sum \sigma_i$ with	gen'd by circles

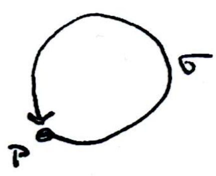


$H_1(\lvert G\rvert)$	$\mathcal{C}_{top}(G)$
Differences:	
• finite sums	• ∞ (thin) sums
• σ not (necessarily) injective	• circles injective
• distinguish only up to 2-boundaries	• remember only the edge sets

6)

Comparing $H_1(|G|)$ with $\mathcal{C}(G)$

Circles are images of 1-simplices σ with $\partial_1 \sigma = 0$, i.e. $\sigma \in \ker \partial_1$.



$\rightarrow$ Does this "correspondence" extend to one between (sums of) circuits and homology classes $[\sigma]$,

$\hookrightarrow$ edge sets of circles

perhaps to a (canonical) group isomorphism

$$f : H_1 \rightarrow \mathcal{C} \;?$$

<u>Task:</u> homology class $\longmapsto$ edge set

$\sigma+\sigma'$ <u>Idea:</u> For each edge $e \in G$, count how often the 1-simplices in $\varphi = \sum \sigma_i$ traverse e

$\uparrow$ coeff's $\in \mathbb{Z}_2$ suppressed

and map $[\varphi] \mapsto \{e \mid \text{this \# is odd}\}$

Realistic?

Problems:

→ f well defined?

- does the count depend on how we concatenate the 1-simplices σ_i (Euler)?
- does it depend on φ itself (vs. on $[\varphi]$)?
- is it always finite?

→ Is f a homomorphism?

→ Is $\operatorname{Im} f \subseteq \mathcal{C}$?

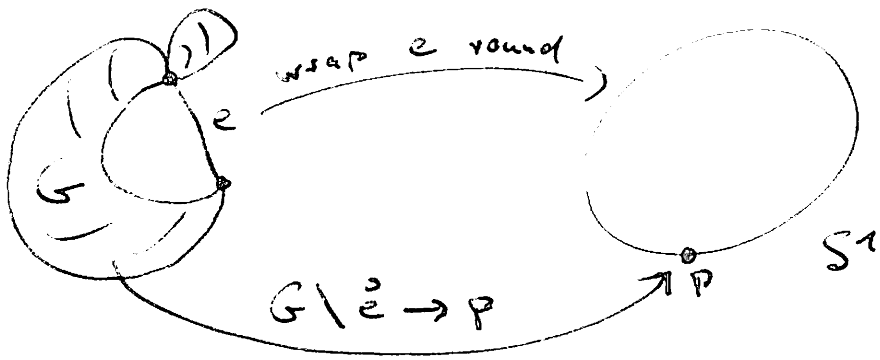
→ Is f surjective (onto $\mathcal{C}$)?

→ Is f injective?

8)

Formal def of f:

For each edge $e \in G$, define map f_e:



f_e is continuous, so $\forall$ simplex σ in $|G|$, $f_e \circ \sigma$ is a simplex in S^1. Moreover,

$$(f_e)_* : \left[\Sigma \sigma_i\right]_{|G|} \longmapsto \left[\Sigma f_e \circ \sigma_i\right]_{S^1}$$

is a (well defined) homomorphism $H_1(|G|) \to H_1(S^1)$.

Using the ("winding #") isomorphism $\pi : H_1(S^1) \to \mathbb{Z}_2$ set

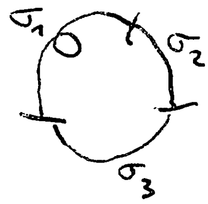
$$f(h) := \{ e \mid (\pi \circ (f_e)_*)(h) = 1 \} \in \underset{\text{edge space}}{\mathcal{E}(G)}$$

Since $\pi \circ (f_e)_*$ is a well-defined group homomorphism so is f.

9)

$\operatorname{Im} f \subseteq \mathcal{C}$:

Want to show : $f([\varphi]) \in \mathcal{C} \quad \forall \varphi \in \operatorname{Ke} \partial_1$

f homom $\Rightarrow$ may assume $\varphi = \sum$ 

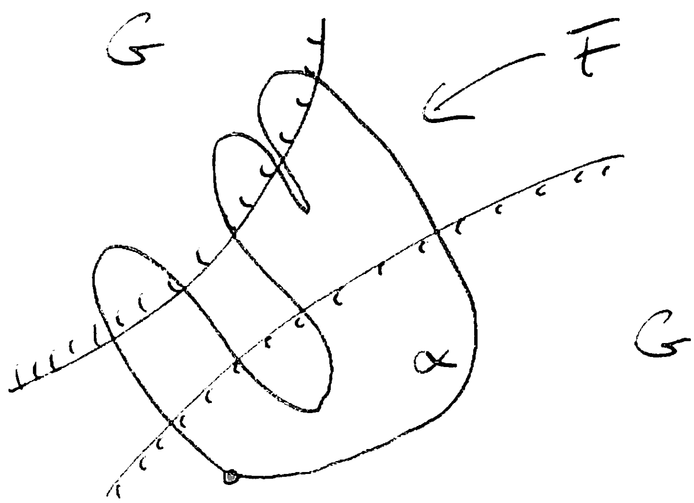
f well def'd $\Rightarrow$ may reparametrize / unsubdivide,

ie assume $\varphi =$ single loop α

Cut criterion for "$\in \mathcal{C}$"

$\Rightarrow$ suff. to show that α crosses for every finite cut F an even $\#$ edges of F odd-often ($\Rightarrow e \in f([\alpha])$)

But that's trivial:



$\square$

10)

f is surjective:

Given $D \in \mathcal{C}$, we find a single loop α s.t. $f([\alpha]) = D$

By def, D is a thin sum of circuits, $C_1, C_2, \ldots$.

Step 0: Pick an NST T; let α_0 be a loop that traverses every edge of T twice and traverses no chord, and pauses at every vertex.

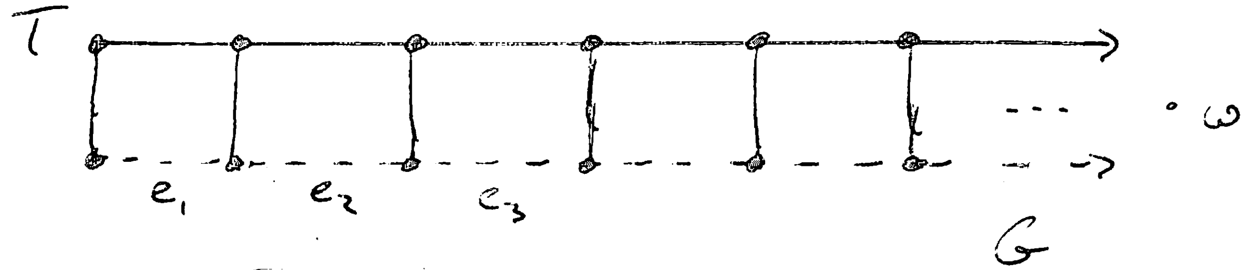
Step n: Insert a tour round C_n at a pause of α_{n-1} at a vertex of C_n to define α_n (again: pause at every vx)

Finally: Define limit α, show it's cont^s.
α traverses every e as often as $2E(T) + C_1 + C_2 + \ldots$

$\Rightarrow f([\alpha]) = D$ $\square$

f is not injective !! (except when $G = \overset{\text{finite}}{G_0} \cup \overset{\text{tree}}{T}$)

Exemple :



α : any loop traversing $\overleftarrow{e}_1 \overleftarrow{e}_2 \overleftarrow{e}_3 \ldots \overrightarrow{e}_1 \overrightarrow{e}_2 \overrightarrow{e}_3 \ldots$ in order

traverses every edge of G an even # times,

so $f([\alpha]) = \emptyset \in \mathcal{E}$, i.e. $[\alpha] \in \ker f$

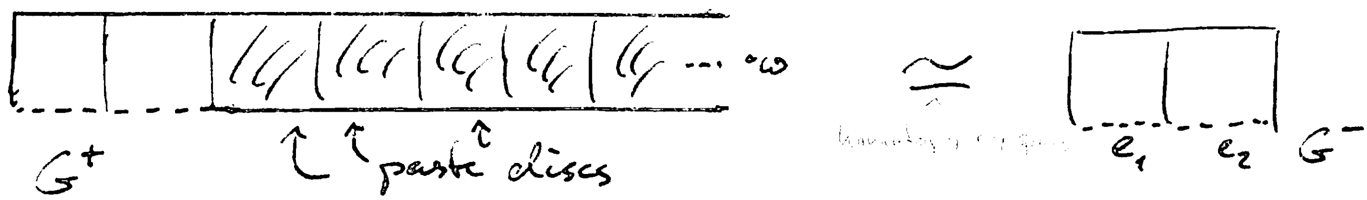
But $[\alpha] \neq 0 \in H_1(|G|)$, i.e. we'll show that

$\alpha \neq \partial\tau_1 + \dots + \partial\tau_n$ $\forall$ 2-simplices $\tau_1 \dots \tau_n$ in $|G|$.

12)

Easy (but weaker): α is not null-homotopic

Pf: any homotopy $\alpha \to$ const in $|G|$
can also be performed in

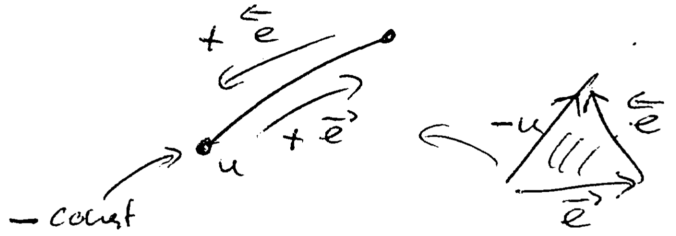


and $\pi_1(G^-) = \langle e_1, e_2 \mid - \rangle = F(2)$

In G^-, $\langle\alpha\rangle = (\underbrace{\overleftarrow{e}_1 \overleftarrow{e}_2 \overrightarrow{e}_1 \overrightarrow{e}_2}_{\text{irreducible word}}) \neq 1$ $\square$

But, of course, $\alpha|_{G^-}$ is null-homologous in G^-:

- subdivide α into edge-passes (two $\forall$ edge!) $\Uparrow$ $[\alpha] \in \operatorname{Ker} f$
- pair these up as $\sum_e (\overrightarrow{e} + \overleftarrow{e})$
- add constant simplices $\to$ Boundaries: $\sum_e (\overrightarrow{e} + \overleftarrow{e} - \text{const})$



13)

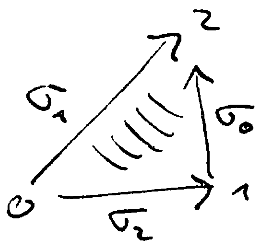
$\nexists$ analogous proof that α is null-homologous in G, since we can subdivide α only finitely often.

Proof that <u>really</u> $[\alpha] \neq 0$, i.e. that

$$\alpha \neq \sum_{i=1}^{n} \partial \tau_i \qquad \forall \text{ 2-simplices } \tau_1, \dots, \tau_n \text{ in } |G|$$

must use some property of these τ !

Namely:



$$\sigma_1 \sim \sigma_2 \sigma_0$$

$\Rightarrow$ <u>want</u>: combinatorial characterization of $\pi_1(|G|)$!

Combinatorial char^n of $\pi_1(|G|)$

Choose a NST T; orient its chords: $\vec{e}_1, \vec{e}_2, \dots$

<u>G finite</u>: $\pi_1(G) = \langle \vec{e}_i \mid - \rangle$

$$\begin{array}{c} \ni \\ \langle \gamma \rangle \end{array} \equiv \text{reduced finite words in the } \vec{e}_i$$

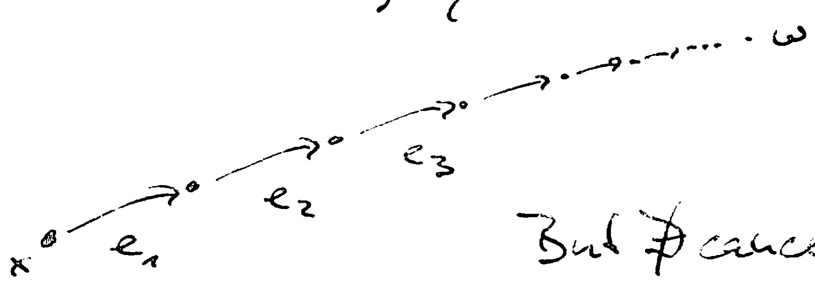
Indeed, $\beta \sim \gamma \iff w(\beta)$ & $w(\gamma)$ reduce to same reduced word

$\downarrow \quad \downarrow$

"traces of β, γ in the chords"

<u>G infinite</u>: traces $w(\gamma)$ are linear orders of $\vec{e}_i$'s of any order type (eg of $\mathbb{Q}$ = wild circle)

<u>Example</u>: the path $\gamma : x \to \omega \to x$ is null-homotopic so its trace $\vec{e}_1 \vec{e}_2 \vec{e}_3 \dots \dots \overleftarrow{e}_3 \, \overleftarrow{e}_2 \, \overleftarrow{e}_1$ <u>should</u> reduce to the empty word:



But $\nexists$ cancelling pair ($\vec{e}_i \overleftarrow{e}_i$).

15)

Solution: we do not define "transfinite reductions" recursively (eg, cancel pairs of letters, pairs of ω-sequences of letters, pairs of ...) but "by compactness":

an infinite word $w = (l_q)_{q \in \mathbb{Q}}$ is reduced if each letter l_q "becomes eventually permanent", ie remains undeleted in the reductions of all finite words $w_I := w \cap \{\vec{e}_i, \overleftarrow{e}_i \mid i \in I\}$ with $I \subseteq \mathbb{N}$ finite but large enough.

Example: no letter in the "doubled wild arc" (null-homotopic) is ever (for any I) permanent

Thm: Every path in $|G|$ is homotopic to a path with a unique reduced trace*

*) which differ for $\beta \neq \gamma$

Hence, $\pi_1(|G|) \equiv \langle$ reduced $\mathbb{Q}$-type words

With this combinatorial description of $\pi_1(|G|)$ we can prove:

<u>Thm</u> f is injective $\iff$

$$\underbrace{T \text{ has only finitely many chords}}_{\text{indep. of choice of } T}$$

<u>Pf '$\Rightarrow$'</u>: $N(\sigma, k) :=$ # tails $\vec{e}_k \vec{e}_{k+1} \vec{e}_{k+2} \dots$ in $\sigma' \sim \sigma$ with reduced trace

$\forall$ 2-simplex τ: $\underline{\exists k}$: $N(\partial\tau, k) = 0$

$\Rightarrow \forall \varphi \in \operatorname{Im} \partial_2$: $\exists k$: $N(\partial\varphi, k) = 0$

But $N(\alpha, k) = 1 \quad \forall k$.

$\square$

A new homology

X: any loc. fin. CW-complex

$\hat{X}$: its Fr-compactification by ends

<u>n-simplex:</u> continuous map $\Delta^n \to \hat{X}$

mapping vertices of Δ^n to X

<u>n-chain:</u> formal sum $\sum_{i \in I} \lambda_i \sigma_i$ with

$(\sigma_i)_{i \in I}$ "locally finite in X":

- every $x \in \underline{X}$ has nbhd meeting $\mathrm{Im}\, \sigma_i$ for only finitely many i
- ($\Leftrightarrow$) every compact $K \subseteq \underline{X}$ meets $\underline{\mathrm{Im}\, \sigma_i}$ for only finitely many i

Thus, ends "are different":

- may lie in ∞ly many σ_i
- may not be 0-faces of any σ_i

<u>boundaries</u>: ... of simplex as before

($\Rightarrow \partial$ (n-simplex) = finite (n-1)-chain)

... of chains: linearly from $\uparrow$ $\left(\partial \sum_{\dots} := \sum \partial_{\dots}\right)$

Note: ∂ preserves local finiteness of chains, so $\partial_n : C_n \to C_{n-1}$ as desired

<u>n-cycles</u>: <u>not</u> all of $\operatorname{Ker} \partial_n$!

Rather: C'_n : finite n-chains

Z'_n : $\operatorname{Ker} \partial_n \cap C'_n$

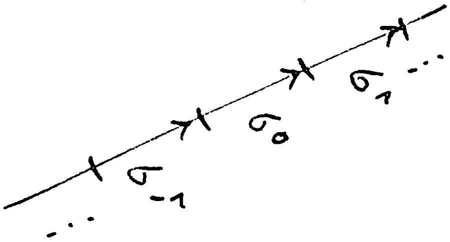
$$Z_n := \{ \varphi \in C_n \mid \varphi = \sum_{j \in J} z_j \text{ with } z_j \in Z'_n \}$$

$$B_n := \operatorname{Im} \partial_{n+1} = \{ \sum \lambda_i \partial_{n+1} \tau_i \mid \tau_i \text{ an } (n+1)\text{-simplex} \}$$

$$\subseteq Z_n, \text{ since } \partial_{n+1} \tau_i \in C'_n$$

$$H_n := Z_n / B_n \quad ; \quad H'_n := \{ [z] \mid z \in Z'_n \} \leq H_n$$

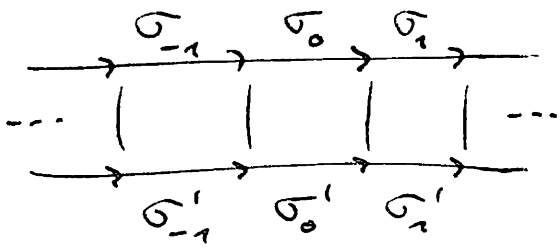
Examples

1. 

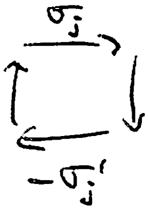
$\sum \sigma_i \in \ker \partial_1 \setminus Z$

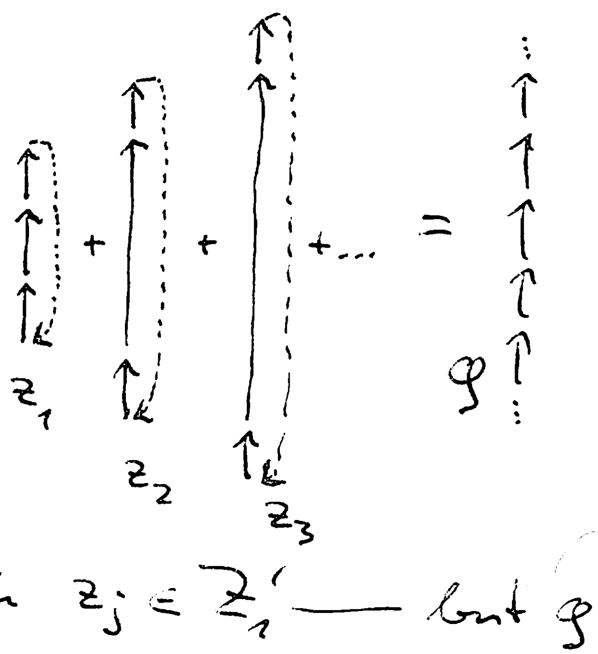
A 1-chain with vanishing boundary that is not a 1-cycle..

... — or is it?

2. 

$\varphi := \sum_{i \in \mathbb{Z}} \sigma_i - \sum_{i \in \mathbb{Z}} \sigma_i' \in Z_1$

since $\varphi = \sum_{j \in \mathbb{Z}} z_j$ with $z_j :=$ 

3. On  consider 

$\hat{=}$ Example 1

$\varphi = \sum_{j \in \mathbb{N}} z_j$ with $z_j \in Z_1'$ — but $\varphi \notin Z_1$

<u>Thm</u>: $H_1 = H_1' \simeq \mathcal{C}$

$\hookrightarrow$ canonically

Pf. Define f "as before" (use that chains are loc. finite)

"$\operatorname{Im} f \subseteq \mathcal{C}$" uses that, by cut criterion (finite cuts!) it suff. to show $f([z_j]) \in \mathcal{C}$ for $z_j \in Z_1'$

"f surjective" as before (since $\alpha \in Z_1'$)

<u>f injective</u>: Given $[\varphi] \in \operatorname{Ker} f$, we can now add an infinite (but locally finite!) chain $b \in B_1$ to φ, "subdividing" φ into a chain $\sum \sigma_i$ with each σ_i traversing <u>one</u> edge only. As $[b] \in \operatorname{Ker} f$, also $\sum \sigma_i \in \operatorname{Ker} f$, so these σ_i pair up  into boundaries. Thus, $[\varphi] \in B_1$.

$\square$

What next?

$$\begin{tikzcd}
\text{Homology} \arrow[r] \arrow[dr] & & \text{Cohomology} \arrow[dl] \\
& \text{flows \& duality } (\mathcal{X}?) &
\end{tikzcd}$$

inf. electrical networks

$\downarrow$

random walks in $|G|$?