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$x^2 + y^3 = z^7$

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(joint work with Edward F. Schaefer and Michael Stoll)

June 8, 2007

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Primitive integer solutions to $x^p + y^q = z^r$

Fix $p, q, r \in \mathbb{Z}_{>0}$. An integer solution (x, y, z) to $x^p + y^q = z^r$ will be called primitive if $\gcd(x, y, z) = 1$. Define

$$\chi := 1/p + 1/q + 1/r - 1.$$

Generalizations of Fermat's *descent* reduce the problem of determining the primitive integer solutions to the determination of the rational points on a finite list of curves (over number fields) whose Euler characteristic $2 - 2g$ is a positive integer multiple of χ. Therefore:

Theorem (Beukers 1998)

If $\chi > 0$, there are infinitely many primitive solutions, coming in finitely many parametrized families.

Theorem (Darmon-Granville 1995 + Faltings 1983 (and Fermat and Euler for $\chi = 0$))

If $\chi \leq 0$, there are at most finitely many primitive solutions.

Known (p, q, r) cases now solved

- $(1, q, r)$
- $(2, 2, n)$
- $(2, 3, n)$ for $n \leq 10$
- $(2, 4, n)$ for $n \leq 8$ and prime $n \geq 211$
- $(2, 2n, 3)$ for prime $7 < n < 10^7$ with $n \neq 31$
- $(2, n, n)$
- $(3, 3, n)$ for $n \leq 6$ and prime $17 \leq n \leq 10000$
- $(3, n, n)$
- $(2n, 2n, 5)$
- (n, n, n)
- permutations of all these except $(2, 3, 10)$, $(2, 4, 7)$, $(2, 2n, 3)$, and $(2, 4, n)$ for prime $n \geq 211$,
- others that reduce immediately to these

Some of the people involved: Bennett, Beukers, Brown, Bruin, Chen, Darmon, Denes, Edwards, Ellenberg, Euler, Fermat, Ghioca, Kraus, Kummer, Lucas, Merel, Mordell, P., Schaefer, Skinner, Stoll, Zagier, based on fundamental work by Breuil, Conrad, Diamond, Frey, Mazur, Ribet, Serre, Shimura, Taylor, Wiles, etc. (this list could be made much longer)

The case $(p, q, r) = (2, 3, 7)$ is of especial difficulty because

- *It achieves the negative value of* χ *closest to* 0, namely
$$1/2 + 1/3 + 1/7 - 1 = -1/42.$$
- *There exist solutions*, some of which are large.
- *The exponents are prime*, so the equation cannot be immediately related to one with smaller exponents. This also prevents solution via elementary factorization arguments, i.e., descent via (geometrically) *abelian* covers. The descent for $(2, 3, 7)$ will involve the simple group of order 168.

Theorem (P.-Schaefer-Stoll)

There are exactly 16 *primitive integer solutions to* $x^2 + y^3 = z^7$*:*

$$(\pm 1, -1, 0), \quad (\pm 1, 0, 1), \quad \pm(0, 1, 1), \quad (\pm 3, -2, 1),$$
$$(\pm 71, -17, 2), \quad (\pm 2213459, 1414, 65),$$
$$(\pm 15312283, 9262, 113), \quad (\pm 21063928, -76271, 17).$$

The highbrow explanation of the (2, 3, 7) descent

(We paraphrase Darmon's explanation of the descent.)

$$\text{primitive integer solutions to } x^2 + y^3 = z^7$$
$$=$$
$$\text{integer points on the scheme}$$
$$S\colon\ \{x^2 + y^3 = z^7\} - \{(0,0,0)\} \text{ in } \mathbb{A}^3_{\mathbb{Z}}.$$

Let's work over $\mathbb{C}$ temporarily:

- $\mathbb{G}_m$ acts on S by $(x, y, z) \mapsto (\lambda^{21}x, \lambda^{14}y, \lambda^6 z)$.
- Stack quotient:
  $[S/\mathbb{G}_m] = \mathbb{P}^1$ with $0, 1, \infty$ replaced by $\frac{1}{2}$-pt, $\frac{1}{3}$-pt, $\frac{1}{7}$-pt.
- $\chi = -1/42 =$ Euler characteristic of this stack.
- Étale covers of $[S/\mathbb{G}_m]$ and hence S can be constructed by finding Galois covers of $\mathbb{P}^1$ with ramification of order $2, 3, 7$ above $0, 1, \infty$.
- The *Riemann Existence Theorem* implies that the Galois group G should be generated by a, b, c satisfying $a^2 = b^3 = c^7 = abc = 1$ (a Hurwitz group).

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(Highbrow explanation, continued)

- The smallest nontrivial Hurwitz group is $G = \mathrm{PSL}_2(\mathbb{F}_7)$ (the simple group of order 168).
- The corresponding étale cover of the stacky $\mathbb{P}^1$ is the Klein quartic

$$X\colon\ x^3y + y^3z + z^3x = 0 \quad \text{in } \mathbb{P}^2.$$

  In fact, this defines an étale cover over $\mathbb{Z}[1/42]$.
- Descent reduces the original problem to finding the $\mathbb{Q}$-points on twists of X by cocycles unramified outside $2, 3, 7$. By Hermite, there are *finitely many* such twists.

Thus the remainder of the proof consists of the following:

1. Find the relevant twists.
2. Find the rational points on these twists.

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Step 1: Finding the relevant twists

We use modularity: $X \to \mathbb{P}^1$ is the same as $X(7) \to X(1)$.

- Each twist of $X(7)$ parametrizes elliptic curves with a nonstandard level-7 structure.
- Each solution (a, b, c) to the original equation gives rise to a "Frey curve" $E_{(a,b,c)}$ with rather special (but not impossible) 7-torsion, and hence a rational point on a special twist as above.

Case 1a: Suppose that $E_{(a,b,c)}[7]$ is reducible.

- Then the element of $H^1(G_{\mathbb{Q}}, \mathrm{PSL}_2(\mathbb{F}_7))$ classifying the twist comes from $H^1(G_{\mathbb{Q}}, B)$ for the Borel subgroup $B = \Gamma_0(7)/\Gamma(7)$ (nonabelian of order 21).
- Since B is a semidirect product, we can construct each such twist in two stages, twisting by a cyclic group each time.
- Since the action on B on the Klein quartic X is known explicitly, these twists may be constructed explicitly by Galois descent.

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Case 1b: Suppose that $E_{(a,b,c)}[7]$ is irreducible.

- By modularity, there is a newform f associated to $E_{(a,b,c)}$.
- Ribet's level lowering shows that if $E_{(a,b,c)}[7]$ is irreducible, then "$f \equiv f' \pmod 7$" for some weight-2 newform f' on $\Gamma_0(N)$ with $N \mid 2^6 3^3$ (up to quadratic twist).
- Stein's tables show that each f' is a quadratic twist of one of 14 newforms f'', of which 13 have coefficients in $\mathbb{Z}$.
- The 14$^{\text{th}}$ has coefficients in $\mathbb{Z}[\sqrt{13}]$, in which 7 is inert, and cannot be congruent mod 7 to a newform with coefficients in $\mathbb{Z}$.
- Thus $E_{(a,b,c)}[7] \simeq E[7]$ where E is one of the 13 curves $24A1, \ldots, 864C1$ (up to quadratic twist).

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- Recall: $X(7)$ is the smooth projective model of the $\mathbb{Q}$-variety $Y(7)$ representing the functor

$$S \mapsto \{(E', \phi) : E'/S \text{ elliptic}, \ \phi\colon \mu_7 \times \mathbb{Z}/7\mathbb{Z} \overset{\wedge}{\simeq} E'[7]\}$$

  where the $\overset{\wedge}{\simeq}$ indicates an isomorphism such that $\bigwedge^2 \phi\colon \mu_7 \to \mu_7$ (using the Weil pairing on the right) is the identity.
- Given $E/\mathbb{Q}$, define the twist $X_E(7)$ as the smooth projective model of $Y_E(7)$ representing

$$S \mapsto \{(E', \phi) : E'/S \text{ elliptic}, \ \phi\colon E[7] \overset{\wedge}{\simeq} E'[7]\}.$$

- For each $a \in (\mathbb{Z}/7\mathbb{Z})^\times$, there is another twist $X_E^a(7)$ defined as for $X_E(7)$, but for which ϕ transforms the Weil pairing on E to the a^{th} power of the Weil pairing on E'.
- The isomorphism type of $X_E^a(7)$ is unchanged if a is multiplied by a square, so as a varies we get only two curves, which we call $X_E(7)$ and $X_E^-(7)$.

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- Each twist of $X(7)$ is a non-hyperelliptic genus-3 curve over $\mathbb{Q}$, and hence is given as $F(x,y,z) = 0$ for some degree-4 form F.
- For $E\colon y^2 = x^3 + ax + b$, an equation for $X_E(7)$ (a form $F(x,y,z)$ with coefficients in $\mathbb{Z}[a,b]$) was given by Halberstadt and Kraus.
- Then we noticed that Salmon's 1879 *Treatise on the higher plane curves* gives an order 4 contravariant Ψ_{-4} of ternary quartic forms; we conjectured and proved that when it is evaluated at the equation of $X_E(7)$, it gives $X_E^-(7)$.

Thus we can write down $X_E(7)$ and $X_E^-(7)$ for each of the 13 elliptic curves over $\mathbb{Q}$.

Step 1, continued: maps to $\mathbb{P}^1$

We need explicit equations not only for the twists of $X(7)$, but also for their degree-168 maps to $\mathbb{P}^1$ given by the j-invariant, so that given points on these twists, we can compute the associated j-invariants and hence the associated primitive solutions to $x^2 + y^3 = z^7$.

- To find the maps, we exploit the fact that they are $\mathrm{PSL}_2(\mathbb{F}_7)$-invariant.
- Specifically, we construct them as ratios of covariants of ternary quartic forms.
- If $F = 0$ is the equation of a twist $X(7)'$ in $\mathbb{P}^2$, then the map is

$$\begin{aligned} X(7)' &\longrightarrow \mathbb{P}^1 \\ (x : y : z) &\longmapsto \frac{\Psi_{14}(F)^3}{\Psi_0(F)\,\Psi_6(F)^7}, \end{aligned}$$

where the Ψ_i are covariants.

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Step 1, continued: the local test

- For each of the finitely many twists constructed, we check whether for every prime p it has $\mathbb{Q}_p$-points that give rise to $\mathbb{Z}_p$-points on S; if not, it gives no primitive integer solutions to $x^2 + y^3 = z^7$ so we discard it.
- We are left with 10 genus-3 curves whose rational points we must find.

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The 10 genus-3 curves

$$C_1: 6x^3y + y^3z + z^3x = 0$$

$$C_2: 3x^3y + y^3z + 2z^3x = 0$$

$$C_3: 3x^3y + 2y^3z + z^3x = 0$$

$$C_4: 7x^3z + 3x^2y^2 - 3xyz^2 + y^3z - z^4 = 0$$

$$C_5: -2x^3y - 2x^3z + 6x^2yz + 3xy^3 - 9xy^2z + 3xyz^2 - xz^3 + 3y^3z - yz^3 = 0$$

$$C_6: x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + 18xyz^2 + 9y^2z^2 - 9z^4 = 0$$

$$C_7: -3x^4 - 6x^3z + 6x^2y^2 - 6x^2yz + 15x^2z^2 - 4xy^3 - 6xyz^2 - 4xz^3 + 6y^2z^2 - 6yz^3 = 0$$

$$C_8: 2x^4 - x^3y - 12x^2y^2 + 3x^2z^2 - 5xy^3 - 6xy^2z + 2xz^3 - 2y^4 + 6y^3z + 3y^2z^2 + 2yz^3 = 0$$

$$C_9: 2x^4 + 4x^3y - 4x^3z - 3x^2y^2 - 6x^2yz + 6x^2z^2 - xy^3 - 6xyz^2 - 2y^4 + 2y^3z - 3y^2z^2 + 6yz^3 = 0$$

$$C_{10}: x^3y - x^3z + 3x^2z^2 + 3xy^2z + 3xyz^2 + 3xz^3 - y^4 + y^3z + 3y^2z^2 - 12yz^3 + 3z^4 = 0$$

Example

The rational point $(0, 1, 1)$ on C_7 gives rise to

$$21063928^2 + (-76271)^3 = 17^7.$$

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Step 2: Determining $C_i(\mathbb{Q})$

Theorem (Faltings 1983, reproved by Vojta 1991)

If X is a curve of genus ≥ 2 over a number field k, then $X(k)$ is finite.

- With work, the proofs of Faltings and Vojta give an upper bound on $\#X(k)$, but this does not let one compute $X(k)$, even in principle.
- In fact, no current algorithm is known to determine $X(k)$ in general, even for genus-2 curves over $\mathbb{Q}$.
- Nevertheless, there are methods, independent of the proofs of Faltings and Vojta, that sometimes succeed for individual curves.

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Let J_i be the Jacobian of C_i.

Step 2a: Determine the rank of $J_i(\mathbb{Q})$.

- The rank is determined by 2-descent, a 2-Selmer group computation.
- It is not yet known how in practice to compute 2-Selmer groups of general genus-3 Jacobians: the most obvious methods require the class group of a number field obtained by adjoining the coordinates of at least one point of $J[2]$, but such a number field is generically of degree 63. (There is, however, work in progress by Bruin, Flynn, P., and Stoll, showing that one can get by with degree-28 class groups.)
- So we developed a method especially for twists of $X(7)$: the geometry of $X(7)$ shows that the Galois action on $J_i[2]$ looks like the Galois action on the 2-torsion of a hyperelliptic genus-3 Jacobian. Then only degree-8 class groups are required.

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Step 2b: Use Chabauty's method to determine $C_i(\mathbb{Q})$ for $i \neq 5$

By adapting Skolem's p-adic method for solving S-unit equations, Chabauty proved

Theorem (Chabauty 1941)

Let X be a curve of genus g over a number field k. Let $J = \operatorname{Jac} X$. If $\operatorname{rank} J(k) < g$, then $X(k)$ is finite.

- Coleman and others showed how to refine this into an effective method for determining $X(k)$, when $J(k)$ is known.
- For $i \neq 5$, we have $\operatorname{rank} J_i(\mathbb{Q}) < 3$ and Chabauty's method determines $C_i(\mathbb{Q})$.
- For $i = 5$, we have $\operatorname{rank} J_5(\mathbb{Q}) = 3$ and Chabauty's method gives no information.

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Step 2b': Use the Brauer-Manin obstruction (sieving out residue classes) to attempt to determine $C_5(\mathbb{Q})$

- Let $C = C_5$ and $J = J_5$.
- Embed C in J.
- It is hard to determine which points of $J(\mathbb{Q})$ lie on C.
- But for a prime p of good reduction, we can determine the subset of points of $J(\mathbb{Q})$ whose image in $J(\mathbb{F}_p)$ lies in $C(\mathbb{F}_p)$. (It will be a union of cosets of a finite-index subgroup of $J(\mathbb{Q})$.)
- If the intersection of these subsets over several p is empty, then we know that $C(\mathbb{Q})$ is empty. (This turns out to be a special case of the Brauer-Manin obstruction, modulo finiteness of $Ш(J)$.)

$$\begin{array}{ccc} C(\mathbb{Q}) & \dashrightarrow & \prod_{p\in S} C(\mathbb{F}_p) \\ \vdots & & \downarrow \\ J(\mathbb{Q}) & \longrightarrow & \prod_{p\in S} J(\mathbb{F}_p). \end{array}$$

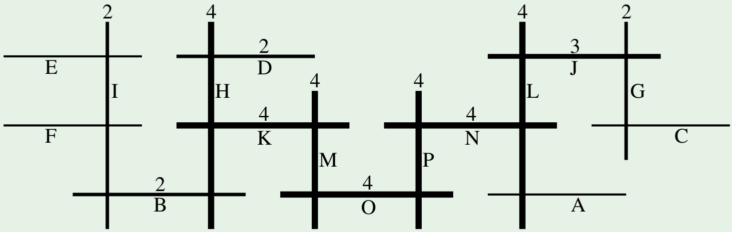
- This doesn't work, since $C(\mathbb{Q})$ is nonempty.

In fact, even today we still don't know $C(\mathbb{Q})$. We got around this problem as follows:

- Points in $C(\mathbb{Q})$ give rise to solutions that are primitive away from 2 and 3, but there are 2-adic and 3-adic conditions that must be satisfied to obtain truly primitive solutions.
- Thus we need only determine the points in $C(\mathbb{Q})$ satisfying these conditions.
- We show that there are none, by incorporating these conditions into the sieve on the previous slide.
- Since $p = 2$ and $p = 3$ are bad for C, in the sieve we must replace $C(\mathbb{F}_p) \hookrightarrow J(\mathbb{F}_p)$ by $\mathcal{C}^{\text{smooth}}(\mathbb{F}_p) \hookrightarrow \mathcal{J}(\mathbb{F}_p)$, where $\mathcal{C}^{\text{smooth}}$ is the smooth locus of the minimal proper regular model of C at p, and $\mathcal{J}$ is the Néron model of J.

Example

For $p = 2$, after iteratively blowing up the initial model eight times, one finds that the special fiber at 2 of the minimal proper regular model of C_5 is



- Combining the sieve information from the bad primes 2 and 3 with the sieve information from the good primes 13, 23, and 97, one rules out rational points in the relevant 2-adic and 3-adic regions.
- This completes the proof. □

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$x^2 + y^3 = z^p$ for $p > 7$?

Our approach generalizes to reduce the study of $x^2 + y^3 = z^p$ for $p > 7$ to the determination of rational points on twists of $X(p)$.

Some steps become easier, but others become harder.

Each solution gives rise to a Frey curve E as before.

Case 1: Reducible $E[p]$.

- The reducible $E[p]$ case becomes almost trivial for $p > 7$ with $p \neq 13$, since there are only finitely many j-invariants of elliptic curves over $\mathbb{Q}$ with reducible $E[p]$ (and none at all for $p > 163$).
- The reducible $E[13]$ case should also be easy: one can reduce to studying rational points on a finite list of twists of the genus-2 curve $X_1(13)$.

Case 2: Irreducible $E[p]$.

- Modularity and level lowering apply as before.
- In fact, the 14 newforms are the same as before.
- The 14^{th} newform can be excluded for all $p \neq 13$ using a method I learned from a paper by Calegari: a given newform with non-integral coefficients can be congruent mod p to a newform with integral coefficients only for a finite, effectively determinable list of p.
- Hence one reduces to determining $X_E(p)$ and $X_E^-(p)$ for the same 13 elliptic curves E as before (plus a problem with the 14^{th} newform if $p = 13$).
- This may be difficult, however, since the genus is much larger (already $g = 26$ for $p = 11$), and again some of these curves have relevant points.

Example

For any p, we have the primitive solution $3^2 + (-2)^3 = 1^p$, associated to $E = 864B1$.