# GLOBAL DIVISIBILITY OF HEEGNER POINTS AND TAMAGAWA NUMBERS

DIMITAR JETCHEV

Let $E/\mathbb{Q}$ be non-CM of conductor $N$. Let $K = \mathbb{Q}(\sqrt{-D})$ with $D \neq 3, 4$ such that all prime divisors of $N$ is split in $K$. Then $N\mathcal{O}_K = \mathcal{N}\bar{\mathcal{N}}$ with $\mathcal{O}_K/\mathcal{N} \simeq \mathbb{Z}/N\mathbb{Z}$. Let $x_1 := [\mathbb{C}/\mathcal{O}_K \to \mathbb{C}/\mathcal{N}^{-1}] \in X_0(N)(H)$, where $H$ is the Hilbert class field of $K$. Let $\phi \colon X_0(N) \to E$ be an optimal modular parameterization (sending $\infty$ to 0). Let $y_K := \operatorname{Tr}_{H/K} \phi(x_1) \in E(K)$; this is a Heegner point.

Gross-Zagier: $y_K$ has infinite order if and only if $L'(E/K, 1) \neq 0$.

Kolyvagin: If $y_K$ has infinite order, then $E(K)$ has rank 1 and $Ш(E/K)$ is finite.

The BSD conjectural formula is equivalent to

$$\#Ш(E/K) = \left(\frac{[E(K) : \mathbb{Z}y_K]}{c\prod_{q|N} c_q}\right)^2$$

where $c$ is the Manin constant.

Choose a prime $p$ such that $p \nmid ND$ and the mod $p$ Galois representation $\rho_{E,p}$ is surjective. Kolyvagin proves that $\#Ш(E/K)[p^\infty] = p^{2(m_0 - m_\infty)}$ where $m_0 = \operatorname{ord}_p([E(K) : \mathbb{Z}y_K])$ and $m_\infty$ is defined in terms of global divisibility of various Heegner points.

Abbes-Ullmo: If $p \nmid N$, then $p \nmid c$.

$p$-part BSD: $m_\infty = \operatorname{ord}_p\left(\prod_{q|N}\right)$.

**Theorem 0.1** (Jetchev)**.** *Let* $m_{\max} = \max_{q|N} \operatorname{ord}_p(c_q)$. *Then* $m_\infty \geq m_{\max}$.

**Example 0.2.** If $E$ has prime conductor, then $\#Ш(E/K)[p^\infty] \leq p^{2(m_0 - m_\infty)}$.

If $E$ has one Tamagawa number divisible by $p$, we get the correct upper bound.

If you try to check the BSD formula for all elliptic curves of conductor $\leq 1000$, there were 200 problematic cases, and William Stein has been able to handle 185 of them using these new results.

C. Cornut: If $r_{\text{an}} = 2$, then $\operatorname{corank}_{\mathbb{Z}_p} \operatorname{Sel}_{p^\infty} \geq 2$.

Suppose $c \nmid N$, and $\mathcal{O}_c = \mathbb{Z} + c\mathcal{O}_K$. Let $\mathcal{N}_c = \mathcal{N} \cap \mathcal{O}_c$, which is an invertible ideal of $\mathcal{O}_c$. Let $x_c = [\mathbb{C}/\mathcal{O}_c \to \mathbb{C}/\mathcal{N}_c^{-1}] \in X_0(N)(K[c])$, where $K[c]$ is the ring class field of conductor $c$. We have $\operatorname{Pic}(\mathcal{O}_c) \simeq \operatorname{Gal}(K[c]/K)$. Let $y_c = \phi(x_c) \in E(K[c])$. "Kolyvagin derivative operators" map $y_c$ to some $P_c \in E(K[c])$, which gives a class $\kappa_{c,m} \in H^1(K, E[p^m])$.

Kolyvagin: Say that $\ell$ is a *Kolyvagin prime* if

- $\ell \nmid pND$
- $\ell$ is inert in $K$
- $p \mid a_\ell, \ell + 1$

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Suppose that $c$ is a squarefree product of Kolyvagin primes. Let $M(c) = \min_{\ell|c} M(\ell)$, where $M(\ell)$ is the exponent of the largest power of $p$ dividing $a_\ell$ and $\ell+1$. Then $\kappa_{c,m}$ works only if $m \le M(c)$.

Let $\Lambda^r$ be the set of all $c$ that are the product of $r$ distinct Kolyvagin primes. Let $\Lambda = \bigcup_r \Lambda_r$. Let $m(c)$ be $\infty$ if $P_c$ is torsion, the largest $m$ such that $P_c \in p^m E(K[c])$ if $m \le M(c)$, and $\infty$ otherwise. Define $m_\infty := \lim_{r\to\infty} \inf_{c\in\Lambda^r} m(c)$.

Property of $\kappa_{c,m}$: If $c \in \Lambda$ and $m(c) + m \le M(c)$, then $\kappa_{c,m}$ is contained in a free $\mathbb{Z}/p^m\mathbb{Z}$-module of rank 1; let $\tilde{\kappa}_{c,m}$ be a generator of this module.

## 1. Local Selmer conditions

Fix $i_v \colon \overline{K} \hookrightarrow \overline{K}_v$ for every place $v$ of $K$. Define the *unramified* condition by

$$H^1_{\mathrm{ur}}(K_v, E[p^m]) := \ker\left(H^1(K_v, E[p^m]) \to H^1(K_v^{\mathrm{ur}}, E[p^m])\right)$$

and the *transverse* condition by

$$H^1_{\mathrm{tr}}(K_\lambda, E[p^m]) := \ker\left(H^1(K_\lambda, E[p^m]) \to H^1(K[\ell]_\lambda, E[p^m])\right)$$

if $\lambda|\ell$. Define the *Kummer* condition by

$$H^1_{\mathrm{Kum}} := \mathrm{image}\left(E(K_v) \to E(K_v)/p^m \overset{\delta}{\hookrightarrow} H^1(K_v, E[p^m])\right).$$

Define the *stringent Kummer* condition by

$$H^1_{\mathrm{Kum}^0} := \mathrm{image}\left(E^0(K_v) \to E(K_v) \to H^1(K_v, E[p^m])\right);$$

this is strictly smaller than $H^1_{\mathrm{Kum}}$ for $v|N$ such that $p|c_v$.

**Proposition 1.1.** *For $v \mid N$, we have* $\mathrm{loc}_v(\kappa_{c,m}) \in H^1_{\mathrm{Kum}^0}(K_v, E[p^m])$.

*Proof.* Use explicit $\kappa_{c,m}$, and the Deligne-Rapoport integral model of $X_0(N)$. Up to a rational torsion point, $y_c \in E^0(K_v)$. □

## 2. Selmer modules

Fix a *Selmer structure*, i.e., a set $\{H^1_{\mathcal{F}}(K_v)\}$ such that $H^1_{\mathcal{F}}(K_v) = H^1_{\mathrm{ur}}(K_v)$ for all but finitely many $v$. We take $H^1_{\mathcal{F}(c)}(K_v) = H^1_{\mathrm{tr}}$. Let $\mathcal{F}^*$ be the dual Selmer structure, in which each subgroup is replaced by its dual under local Tate duality. Define

$$H^1_{\mathcal{F}}(K; E[p]) := \ker\left(H^1(K, E[p^m]) \to \bigoplus H^1(K_v)/H^1_{\mathcal{F}}(K_v)\right).$$

*Proof of Theorem.* We want to show $m_\infty \ge m_{\max}$; i.e., that $p^{m_{\max}} \mid \kappa_{c,m}$.

(1) Reduce to an easy case for $c$ with $m(c) = m_\infty$, $m(c) + m \le M(c)$, $H^1_{\mathcal{F}_{\mathrm{Kum}}(c)} \simeq \mathbb{Z}/p^m\mathbb{Z}$.
(2) Use Chebotarev density to choose a suitable $\ell$ such that the length of the module $H^1_{\mathcal{F}_{\mathrm{Kum}}(c\ell)}$ is small, $\le m - m_{\max}$.

□