p-ADIC LANGLANDS: LOCAL-GLOBAL COMPATIBILITY

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Let E be a finite extension of $\mathbb{Q}_p$, with valuation ring $\mathcal{O} = \mathcal{O}_E$ and residue field k.

Let $\bar{\rho}\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(k)$ be unramified outside p, continuous, absolutely irreducible, modular (odd).

Let $\operatorname{Spf} R$ be the universal deformation space for lifts of $\bar{\rho}$ unramified outside p. This is a formal scheme over $\operatorname{Spf} \mathcal{O}$. The generic fiber of this space is 3-dimensional. Inside it are points that come from modular forms. Problem: How do you recognize these points (i.e., recognize modular lifts of $\bar{\rho}$ among all lifts)?

p-adic local Langlands (Breuil, Berger, Colmez): Consider

$$\{\text{f.g. smooth admissible } \mathrm{GL}_2(\mathbb{Q}_p)\text{-representations over a local Artinian } \mathcal{O}\text{-algebra } A\}$$

(*smooth* means that every vector is fixed by some congruence subgroup). Colmez's functor MF goes from this to

$$\{\text{f.g. } A\text{-module with } G_{\mathbb{Q}_p}\text{-action}\}.$$

Inside $D_p := \mathrm{GL}_2(\mathbb{Q}_p)$ is the $ax+b$ group $\begin{pmatrix} * & * \\ 0 * 1 & \end{pmatrix}$; the restriction of the representations to this subgroup correspond to (ϕ, Γ)-modules.

Let ω be the mod p cyclotomic character.

Theorem 0.1 (Colmez, Emerton)**.** *If $\bar{\rho}|_{D_p} \not\simeq$ twist $\otimes \begin{pmatrix} 1 & * \\ 0 * \omega & \end{pmatrix}$, then there exists a unique representation $\bar{\pi}$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ over k mapping via MF to $\bar{\rho}$.*

p-adic local Langlands conjecture (almost proved by Colmez; incorporating a strategy of Kisin):

1. (1) MF induces an equivalence $\mathrm{Def}(\bar{\pi}) \xrightarrow{\sim} \mathrm{Def}(\bar{\rho}|_{D_p}$
2. (2) If ρ_p lifts $\bar{\rho}|_{D_p}$ to characteristic 0, and is potentially semistable with Hodge-Tate weights $(0, k-1)$ for some $k > 1$, then the corresponding π is a completion of $\mathrm{LL}(\mathrm{WD}(\rho_p)^{\text{F-ss}}) \otimes \left(\mathrm{Sym}^{k-2} E^2\right)^{\vee}$. Here WD is the Weil-Deligne representation, and F-ss denotes Frobenius semisimplification, and LL denotes local Langlands.

Granting this, one can form π_R an orthonormalizable R-Banach module with $\mathrm{GL}_2(\mathbb{Q}_p)$-action. Consider $\rho_R \otimes_R \pi_R$ with action of $R[G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{Q}_p)]$.

Let

$$\hat{H}^1_{\bar{\rho}} = \varprojlim_s \varinjlim_r H^1_{\text{et}}(X(p^r), \mathcal{O}/p^s)_{\bar{\rho}}.$$

This is a module with an action of a Hecke algebra $\mathbb{T}[G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{Q}_p)]$.

Have $R \twoheadrightarrow \mathbb{T}$, $\operatorname{Spf} \mathbb{T} \hookrightarrow \operatorname{Spf} R$ (Zariski closure of moduli points).

Conjecture 0.2 (Mazur)**.** $R = \mathbb{T}$.

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This has been proved in many cases by Gouvea-Mazur, Böckle, Emerton-Kisin (if $p > 2$ and $\bar{\rho}|_{G_{\mathbb{Q}(\zeta_p)}}$ is absolutely irreducible).

Theorem 0.3. *Granting (1) and (2) of p-adic local Langlands, so that π_R exists, let*

$$X := \mathrm{Hom}_{R[G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{Q}_p)]}(\rho_R \otimes \pi_R, \hat{H}^1_{\bar{\rho}}).$$

Then

(a) *X is a co-f.g. faithful $\mathbb{T}$-module. That is, $\mathrm{Hom}_{\mathcal{O}}(X, \mathcal{O})$ are f.g. over $\mathbb{T}$.*
(b) *$\rho_{\mathbb{T}} \otimes_{\mathbb{T}} \pi_{\mathbb{T}} \otimes_{\mathbb{T}} X \xrightarrow{\mathrm{ev}} \hat{H}^1_\rho$ is onto if we tensor with $\mathbb{Q}_p$.*
(c) *ev is an isomorphism, and X is cofree of rank 1 over $\mathbb{T}$, provided that $\mathrm{End}\,\bar{\rho}|_{D_p} = k$.*

This proves some cases of the Fontaine-Mazur conjecture.

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