EXOTIC HEEGNER POINTS

HENRI DARMON

Heegner points: Define $\Phi\colon \mathrm{Div}^0(X_0(N)) \to E$ by $\Phi(\Delta) = \int_{C \text{ s.t. } \partial C = \Delta} \omega_f \in \mathbb{C}/\Lambda \simeq E(\mathbb{C})$ where $\omega_f = f(z)\,dz$. We have a collection $\mathrm{CM}_K \subset X_0(N)(K^{\mathrm{ab}})$. Then

$$\{\Phi(\Delta)\}_{\Delta \in \mathrm{Div}^0(\mathrm{CM}_K)} \subseteq E(K^{\mathrm{ab}}).$$

- Kolyvagin-GZ: If $\mathrm{ord}_{s=1} L(E,s) \le 1$, then BSD is true and Ш is finite.
- Efficient algorithms for calculating points in $E(\mathbb{Q})$ and $E(K^{\mathrm{ab}})$.

Question: Can we generalize this structure?

Around 2000, I proposed "Stark-Heegner points".

Prototypical Stark-Heegner construction. Data:

- A Shimura variety X
- A differential r-form ω on X
- A null-homologous $(r-1)$-cycle $\Delta \subset X(\mathbb{C})$

$\{\int_\gamma \omega\} \subseteq \Lambda$ in $\mathbb{C}$ and $\mathbb{C}/\Lambda = E(\mathbb{C})$

Stark-Heegner conjecture: $\int_{C \text{ s.t. } \partial C = \Delta} \omega \in \mathbb{C}/\Lambda \simeq E(\mathbb{C})$ belong to $E(\overline{\mathbb{Q}})$.

Difficulty: We lack a good algebro-geometric understanding of the integrals that appear.

"Exotic Heegner point": $\Delta \leftrightarrow$ algebraic cycles on X.

What follows is joint work with Bertolini and Kartik Prasanna.

1. General setup

Let K be an imaginary quadratic field of class number 1. Let $-D = \mathrm{disc}(K)$. Assume that $D \ne 3, 4, 8$. (There are six remaining possibilities for D.)

There exists $A/\mathbb{Q}$ such that $\mathrm{End}_K(A) \simeq \mathcal{O}_K$; among these, take A so that $\mathrm{cond}(A)$ is minimal; i.e., $\mathrm{cond}(A) = D^2$. This determines A uniquely up to $\mathbb{Q}$-isogeny.

Define

$$L(A/\mathbb{Q}, s) = \sum_{\mathfrak{a} \le \mathcal{O}_K} \psi(\mathfrak{a})(N\mathfrak{a})^{-s}$$

where $\psi((a)) = \epsilon_K(a)a$, where $\epsilon_K\colon (\mathcal{O}_K/\sqrt{D})^\times \to \pm 1$. Let $f = \sum_{\mathfrak{a}} \psi(a) q^{N\mathfrak{a}}$, which is a theta series of weight 2 on $\Gamma_0(D^2)$.

2. Exotic modular parameterizations

Fix $r \ge 0$. Define

$$\theta_{\psi^{r+1}} := \sum \psi(\mathfrak{a})^{r+1} q^{N\mathfrak{a}} \in \begin{cases} S_{r+2}(\Gamma_0(D^2)), & \text{if } r \text{ is even;} \\ S_{r+2}(\Gamma_0(D^2), \epsilon_K), & \text{if } r \text{ is odd.} \end{cases}$$

Let t be 1 if r is odd, and 2 if r is even.

Date: June 6, 2007.

Geometric interpretation of θ_r: Let W_r be the r-fold fiber product of the universal elliptic curve $\mathcal{E} \to X_0(D^t)$. Then $\dim W_r = r+1$. Then $W_r(\mathbb{C})$ contains $(\mathbb{Z}^{2r} \rtimes \Gamma_0(D^t))\backslash(\mathbb{C}^r \times \mathcal{H})$. Let $w_1, \ldots, w_r, \tau$ be the coordinates on $\mathbb{C}^r \times \mathcal{H}$. Let

$$\omega_{\theta_r} := \theta(\tau)\, d\tau\, dw_1 \cdots dw_r \in \Omega^{r+1,0}(W_r).$$

The variety: $X_r := A^r \times W_r$. So $\dim X_r = 1 + 2r$.

The differential form: $\omega_r := \bar{\omega}_A^r \wedge \omega_{\theta^r} \in \Omega^{r+1,r}(X_r)$, where ω_A is a Néron differential on A.

The exotic parametrization: Replace $\mathrm{Div}^0(X_0(D^2))$ by $\mathrm{CH}^{r+1}(X_r)_0$, which is defined as the set of null-homologous codimension-$(r+1)$ algebraic cycles on X_r modulo rational equivalence. Define a complex analytic map

$$\begin{aligned} \Phi_r \colon \mathrm{CH}^{r+1}(X_r)_0 &\to A \\ \Delta &\mapsto \int_{C \text{ s.t. } \partial C = \Delta} \omega_r \in \mathbb{C}/\Lambda_A \simeq A(\mathbb{C}). \end{aligned}$$

Conjecture 2.1. For any field F, if $\Delta \in \mathrm{CH}^{r+1}(X_r)_0(F) \otimes \mathbb{Q}$ (i.e., Δ is defined over F), then $\Phi_r(\Delta) \in A(F) \otimes \mathbb{Q}$.

3. Heegner cycles

We have a natural projection $\pi\colon X_r \to X_0(D^t)$. Then $\pi^{-1}(\tau) \simeq A^r \times E_\tau^r$. Let $\phi\colon A \to A'$ be a cyclic n-isogeny, with $D \nmid n$, defined over the ring class field H_n of conductor n. Let

$$\Delta_\phi = (\mathrm{Graph}(\phi) - (\deg\phi)(0 \times A') - (A \times 0))^r \subseteq (A \times A')^r \to X_r.$$

Then

$$\Delta_\phi \in \begin{cases} \mathrm{CH}^{r+1}(X_r)_0(H_n), & \text{if } r \text{ is odd;} \\ \mathrm{CH}^{r+1}(X_r)_0(H_{nD}), & \text{if } r \text{ is even.} \end{cases}$$

Conjecture 2.1 implies that

$$\Phi_r(\Delta_\phi) \in \begin{cases} A(H_n), & \text{if } r \text{ is odd;} \\ A(H_{nD}), & \text{if } r \text{ is even} \end{cases}$$

up to torsion.

4. Exotic Heegner points

If r is even, then $\mathrm{tr}_{H_n}^{H_{nD}}(\Phi_r(\Delta_\phi)) = 0$.

So assume that r is odd.

Conjecture 4.1. Let $y_r = \Phi_r(\Delta_1)$; i.e., ϕ is the identity $A \to A$, so Conjecture 2.1 implies that $y_r \in A(\mathbb{Q})$ modulo torsion. Then y_r is of infinite order if and only if $L(\psi^{2r+1}, r+1) \neq 0$ and $L'(\psi, 1) \neq 0$. The height of y_r corresponds to a derivative of a Rankin convolution L-series: $L(\theta_{\psi^r} \times \theta_{\psi^{r+1}}, s)_{r+1} = L(\theta_{\psi^{2r+1}}, s) L(\theta_\psi, s-r)$.

5. Evidence for the conjectures

Numerical evidence: $D = 11$, $n = 13$, $A = 121B$, $r = 1$. Get a point of $A(H_{13})$.

Theoretical evidence: p-adic analogues can be formulated and proved in certain cases.

We have cycle class maps

$$\mathrm{cl}\colon \mathrm{CH}^{r+1}(X_r)(\mathbb{Q}) \to H^{2r+2}_{\mathrm{et}}(X_r/\overline{\mathbb{Q}}, \mathbb{Q}_p)(r+1).$$

Let $\Phi_{r,p}$ be the composition

$$\ker(\mathrm{cl}) \to H^1(\mathbb{Q}, H^{2r+1}(X_r/\overline{\mathbb{Q}}, \mathbb{Q}_p)(r+1)) \to H^1(\mathbb{Q}, V_p(A)) \to H^1_f(\mathbb{Q}_p, T_p(A)) \simeq A(\mathbb{Q}_p) \otimes \mathbb{Q}$$

where $V_p(A) = T_p(A) \otimes \mathbb{Q}$.

Theorem 5.1 (Bertolini-Darmon-Prasanna)**.** *Suppose that p splits in K. Then $\Phi_{r,p}(\Delta_1) \in A(\mathbb{Q}) \otimes \mathbb{Q}$, and is independent of p. It is of infinite order if and only if $L(\psi^{2r+1}, r+1) \neq 0$ and $L'(\psi, 1) \neq 0$.*