MODULAR FORMS AND HECKE OPERATORS OVER NUMBER FIELDS

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1. Background

Around 1980, John Cremona in his thesis worked out cusp forms of weight 2 for $\Gamma_0(\mathfrak{n})$ over $\mathbb{Q}(\sqrt{-d})$ for $d = 1, 2, 3, 7, 11$.

Around 1990, his student Whitley in her thesis worked out $d = 19, 43, 67, 163$.

At this point, one could no longer avoid the case of class number greater than 1. In 1999, Bygott handled $d = 5$ (with class number $h = 2$): the tricks worked out here worked for ideal classes whose square was trivial.

In 2005, Mark Lingham worked out $d = 23, 31$ (with $h = 3$). In fact, odd class number turned out to be easier than even class number.

2. Notation

Let K be a number field. Embed $K \hookrightarrow K_\infty = \mathbb{R}$ or $\mathbb{C}$. Let $R = \mathcal{O}_K$ be the ring of integers of K. Let $0 \neq \mathfrak{n} \leq R$ be an ideal. Let $\mathrm{Cl} = \mathrm{Cl}(K)$. Let $h = \#\,\mathrm{Cl} = h_2 h_2'$, where $h_2 = \#\,\mathrm{Cl}[2]$. Let $\Gamma = \mathrm{GL}(2, R)$, acting on $R \oplus R$ on the *right*. Let $\Gamma_0(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \in \mathfrak{n} \right\}$, which is the set of $\gamma \in \Gamma_0(\mathfrak{n})$ such that $\begin{pmatrix} R & R \\ \mathfrak{n} & R \end{pmatrix} \gamma = \begin{pmatrix} R & R \\ \mathfrak{n} & R \end{pmatrix}$. Define

$$\varphi(\mathfrak{n}) = \#(R/\mathfrak{n}) = N(\mathfrak{n}) \prod_{\mathfrak{p} | \mathfrak{n}} (1 - N(\mathfrak{p}))^{-1}$$

$$\psi(n) = [\Gamma : \Gamma_0(N)] = N(\mathfrak{n}) \prod_{\mathfrak{p} | \mathfrak{n}} (1 + N(\mathfrak{p}))^{-1}$$

3. Lattices

A *lattice* is a rank-2 R-submodule L of $K_\infty \oplus K_\infty$ (i.e., contained in $\mathbb{R} \oplus \mathbb{R} \leftrightarrow \mathbb{C}$ or in $\mathbb{C} \oplus \mathbb{C} \leftrightarrow \mathbb{H}$. As an R-module, we have $R \simeq \mathfrak{a} \oplus R$ for some ideal $\mathfrak{a} \leq R$. Define the *Steinitz class* $[L] := [\mathfrak{a}] \in \mathrm{Cl}$.

Lemma 3.1. *If $\mathfrak{a}\mathfrak{b} = \langle g \rangle$, then $R \oplus R \simeq \mathfrak{a} \times \mathfrak{b}$.*

Proof. Write $\mathfrak{a} = \langle a_1, a_2 \rangle$. Write $g = a_1 b_2 - a_2 b_1$ with $b_i \in \mathfrak{b}$. Right multiplication by the matrix $M := \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ gives an isomorphism $R \oplus R \simeq \mathfrak{a} \oplus \mathfrak{b}$. Such matrices are called (a, b)-matrices. □

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Remark 3.2. In choosing M, the entry a_2 (as second generator of $\mathfrak{a}$) may be chosen to be any nonzero element of R.

If M_1, M_2 are both (a,b)-matrices then

- $M_1 = \gamma M_2$ for some $\gamma \in \Gamma$
- $M_1 = M_2\gamma'$ where $\gamma' \in \left\{ \begin{pmatrix} R & \mathfrak{a}^{-1}\mathfrak{b} \\ \mathfrak{a}\mathfrak{b}^{-1} & R \end{pmatrix} : \det \in R^\times \right\} =: \Gamma^{\mathfrak{a},\mathfrak{b}}$.

Applications: cusp equivalence under Γ and $\Gamma_0(\mathfrak{n})$. A cusp is represented by $\alpha = a_1/a_2 \in \mathbb{P}^1(K)$ with $a_i \in R$. The class $[\langle a_1, a_2\rangle] =: [\alpha]$ is well-defined. Two elements of $\mathbb{P}^1(K)$ are in the same Γ-orbit if and only if they have the same class.

Proposition 3.3. *The number of $\Gamma_0(\mathfrak{n})$-orbits of cusps equals $h\sum_{\delta|\mathfrak{n}}\varphi_n(\delta+\mathfrak{n}\delta^{-1})$ where $\varphi_u(\mathfrak{n}) = \#\left((R/\mathfrak{n})^\times/R^\times\right)$ (where the quotient means cokernel in case $R^\times \to (R/\mathfrak{n})^\times$ is not injective).*

Standard lattices: Let $\mathfrak{p}_i$ for $1 \le i \le h_2$ represent $\mathrm{Cl}/\mathrm{Cl}^2$ with $\mathfrak{p}_1 = \langle 1\rangle$. Let $\mathfrak{q}_j$ for $1 \le j \le h_2'$ be such that the $\mathfrak{q}_j^2$ represent Cl^2. Thus $\mathrm{Cl} = \{c_{ij} = [\mathfrak{p}_i\mathfrak{q}_j^2]\}$. Let $L_{ij} = \mathfrak{q}_j(\mathfrak{p}_i \oplus R) = \mathfrak{q}_j\mathfrak{p}_i \oplus \mathfrak{q}_j$. Then $[L_{ij}] = c_{ij}$.

4. Modular points for $\Gamma_0(\mathfrak{n})$

A *modular point* is a pair (L, L') where $L' \supseteq L$ with $L'/L \simeq R/\mathfrak{n}$. *Standard modular points* are $P_{ij} := (L_{ij}, L'_{ij})$ where $L'_{ij} = \mathfrak{q}_j(\mathfrak{p}_i \oplus \mathfrak{n}^{-1})$. Then $G = \mathrm{GL}(2, K_\infty)$ acts on modular points: an element $U \in G$ maps (L, L') to $(LU, L'U)$. Every modular point P is $P_{ij}U$ for some $U \in G$, and we may consider $\Gamma_0^{\mathfrak{p}_i}(\mathfrak{n})\backslash G$ where $\Gamma_0^{\mathfrak{p}_i}(\mathfrak{n}) := \left\{ \begin{pmatrix} R & \mathfrak{p}_i^{-1} \\ \mathfrak{p}_i\mathfrak{n} & R \end{pmatrix} : \det \in R^\times \right\}$. *Formal modular forms* are functions of modular points. These correspond to collections of h functions ϕ_{ij} on G where ϕ_{ij} is left-invariant by $\Gamma_0^{\mathfrak{p}_i}(\mathfrak{n})$.

5. Hecke operators

Let $\mathcal{M}_0(\mathfrak{n})$ be the set of modular points for $\Gamma_0(\mathfrak{n})$. Let $\mathbb{T}$ be the commutative algebra of operators on $\mathbb{Q}\mathcal{M}_0(\mathfrak{n})$.

For $\mathfrak{a} \le \mathfrak{n}$, we have

$$T_{\mathfrak{a}}\colon (L, L') \mapsto N(\mathfrak{a})^{-1} \sum_{\substack{M \supseteq L \\ [M:L]=\mathfrak{a} \\ (M,M')\in\mathcal{M}_0(\mathfrak{n})}} (M, M')$$

where $M' = M + L'$. For $\mathfrak{a}$ coprime to $\mathfrak{n}$, we have

$$T_{\mathfrak{a},\mathfrak{a}}\colon (L, L') \mapsto N(\mathfrak{a})^{-2}(\mathfrak{a}^{-1}L, \mathfrak{a}^{-1}L').$$

We have a formal identity

$$\sum_{0\neq\mathfrak{a}\le R} T_{\mathfrak{a}}N(\mathfrak{a})^{-s} = \prod_{\mathfrak{p}|\mathfrak{n}} \left(1 - T_{\mathfrak{p}}N(\mathfrak{p})^{-s}\right)^{-1} \prod_{\mathfrak{p}\nmid\mathfrak{n}} \left(1 - T_{\mathfrak{p}}N(\mathfrak{p})^{-s} + T_{\mathfrak{p},\mathfrak{p}}N(\mathfrak{p})^{1-2s}\right)^{-1}.$$

Define $[T_{\mathfrak{a}}] = [\mathfrak{a}]$ and $[T_{\mathfrak{a},\mathfrak{a}}] = [\mathfrak{a}]^2$ so $[T_{\mathfrak{a}}T_{\mathfrak{b},\mathfrak{b}}] = [\mathfrak{a}\mathfrak{b}^2]$. Let $\mathbb{T}_c$ be the submodule of $\mathbb{T}$ consisting of operators of class $c \in \mathrm{Cl}$. Then $\mathbb{T} = \oplus_{c\in\mathrm{Cl}}\mathbb{T}_c$ is a *grading*: $\mathbb{T}_{c_1}\mathbb{T}_{c_2} \subset \mathbb{T}_{c_1c_2}$. If (L, L') has class $c = [L]$, and $T \in \mathbb{T}_{c'}$, then $T(L, L')$ has class cc'.

Practical question: To what extent do the *principal* Hecke operators determine the whole Hecke action? Answer: Enough to be useful.

6. Formal modular forms

Functions $\mathcal{M}_0(\mathfrak{n}) \to \mathbb{C}^r$ are in bijection with collections of h functions $\phi_{ij}\colon G \to \mathbb{C}^r$ such that $\phi_{ij}(\gamma u) = \phi_{ij}(u)$ with $\gamma \in \Gamma_0^{\gamma_i}(\mathfrak{n})$. Introduce a further action on the right by $ZK \subset G$, where Z is the center $\mathbb{R}^\times$ or $\mathbb{C}^\times$ and K is $O(2)$ or $U(2)$, via a representation $\rho\colon ZK \to \mathrm{GL}(r, \mathbb{C})$. Write $G = ZBK$ where $B := \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} : x \in K_\infty, y \in \mathbb{R}_{>0} \right\}$, which corresponds to $\mathcal{H}_2$ or $\mathcal{H}_3$.