COUNTING AUTOMORPHIC FORMS

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1. Classical modular forms

Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Let $\Gamma(q)$ be the congruence subgroup. Let $S_2(\Gamma(q))$ be the space of cuspidal modular forms of weight 2 and level $\Gamma(q)$. Let $Y(q) = \Gamma(q)\backslash\mathcal{H}$. Let $X(q)$ be the usual compactification of $Y(q)$. Then

$$\dim S_2(\Gamma(q)) = \text{genus of } X(q) \approx \frac{1}{12}[\Gamma : \Gamma(q)] \approx cv(q)$$

where $v(q)$ is the area of $X(q)$ and $c > 0$ is some constant. Similarly, if $k \geq 2$, then $\dim S_k(\Gamma(q)) \approx c_k v(q)$. What if $k = 1$? The Selberg trace formula implies that

$$\dim S_1(\Gamma(q)) \ll \frac{v(q)}{\log v(q)}.$$

where $A_q \ll B_q$ means $A_q < cB_q$ for some constant c.

Theorem 1.1 (Duke 1995)**.** *$\dim S_1(\Gamma(q)) \ll v(q)^{1-\mu}$ for some explicit μ (e.g., $\mu = 1/36$ is OK).*

What is the difference between $k = 1$ and $k \geq 2$? If f is a cuspidal Hecke eigenform, then one can assign to f an automorphic representation π of $\mathrm{GL}_2(\mathbf{A}_\mathbb{Q})$, and in particular a representation π_∞ of $\mathrm{GL}_2(\mathbb{R})$. If $k \geq 2$, then π_∞ is discrete series. But if $k = 1$, it is a limit of discrete series.

2. Semisimple group

Replace $\mathrm{GL}_2(\mathbb{Q})$ by $\mathbb{G}$ over F, where $\mathbb{G}$ is semisimple. We have $\rho\colon \mathbb{G}(F) \hookrightarrow \mathrm{GL}_n(\mathbb{Q})$ (by restriction of scalars). Let $\mathbb{G}(\mathcal{O}_F) = \rho^{-1}(G(\mathbb{R}) \cap \mathrm{GL}_n(\mathbb{Z}))$. Let Γ be commensurable with $G(\mathcal{O}_F)$. For example, $\Gamma(q) := \rho^{-1}(\rho(\Gamma) \cap q\text{-congruence of } \mathrm{GL}_n(\mathbb{Z}))$.

Example 2.1. Let K be an imaginary quadratic field. Let $\mathbb{G} = \mathrm{GL}_2$ over k. Let $\Gamma = \mathrm{GL}_2\,\mathcal{O}_K$.

3. Flavors of automorphic forms

- cohomological type, or not
- π_∞ discrete series (or more generally tempered), or non-tempered

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	$\mathbb{G}$	π_∞	coh type	$\mathbb{G}(\mathbb{R})/K(\mathbb{R})$
classical MF	$\mathrm{GL}_2/\mathbb{Q}$	disc	yes	$\mathcal{H}_2$
weight $k=1$	$\mathrm{GL}_2/\mathbb{Q}$	temp	no	$\mathcal{H}_2$
Hilbert MF regular wt	GL_2/K, K tot real	disc	yes	$(\mathcal{H}_2)^{r_2}$
Siegel MF $g=2$, (k_1,k_2) with $k_1>k_2\geq 3$	$\mathrm{GSp}_4/\mathbb{Q}$	disc	Sieg	
MF assoc to Shimura var, reg wt	various	disc	yes	
MF GL_2/K, K not tot real	temp, not discrete	yes	$(\mathcal{H}_3)^{r_1}\times(\mathcal{H}_2)^{r_2}$	
MF $\mathrm{GL}_n/\mathbb{Q}$, $n\geq 3$	$\mathrm{GL}_n/\mathbb{Q}$	temp?, not discrete	yes	
$\mathcal{O}(n,1)$, for $n\geq 4$	$\mathcal{O}(n,1)$	non-temp	yes	$\mathcal{H}^n$

4. Counting

Theorem 4.1 (DeGeorge-Wallach). *One gets the "greatest possible" number of automorphic forms if and only if one is in the discrete series case. Define the manifold* $Y(q) := \Gamma(q)\backslash G(\mathbb{R})\rangle K(\mathbb{R})$. *Then* $\dim H^*(Y(q),\nu)$ *is*

$$\begin{cases}\approx cv(q) & \textit{(discrete series)}\\ \ll \frac{v(q)}{\log v(q)} & \textit{(other)}.\end{cases}$$

Theorem 4.2 (Sarnak-Xu). *In the non-tempered case,* $\dim H^*(Y(q),\nu)\ll v(q)^{1-\mu}$ *for some* $\mu>0$.

Conjecture 4.3 (Sarnak-Xu). One can take $1-\mu=\frac{2}{p+\epsilon}$, where

$$p := \inf\{s : \text{matrix coefficients of } \pi_\infty \text{ are in } L^s(G)\}.$$

Theorem 4.4 (Calegari-Emerton). *Suppose* $\mathbb{G}$ *does not admit discrete series. Fix prime* $\mathfrak{p}$ *of* $\mathcal{O}_F$. *Then*

$$\dim H^*(Y(N\mathfrak{p}^k),\nu)\ll_{N,\mathfrak{p}} v(N\mathfrak{p}^k)^{1-\frac{1}{\dim G(\mathbb{R})}}$$

as $k\to\infty$.

Example 4.5. Let K be an imaginary quadratic field. Let $F=\mathrm{GL}_2(\mathcal{O}_K)$. Let $p=\pi\bar{\pi}$. Then $\dim H^1\left(\Gamma(N\bar{\pi}^n)\backslash\mathcal{H}^3,\mathbb{C}\right)\ll p^{2n}$. (The trivial bound is p^{3n}.) If $\dim>0$, then $\gg p^n$. Also, $\dim H^1\left(\Gamma_1(N\pi^n)\backslash\mathcal{H}^3,\mathbb{C}\right)\ll p^n$.

Theorem 4.6 (Calegari-Dunfield). *Let* $K=\mathbb{Q}(\sqrt{-2})$ *and let* $p=3=\pi\bar{\pi}$. *Then*

$$\dim H^1_{cusp}\left(\Gamma(\pi^n)\backslash\mathcal{H}^3,\mathbb{C}\right)=0$$

for all n.

Look up the $\Gamma(\pi^n)$-tower of $Y_n := Y(N\pi^n)$. The p-adic analytic group $G=\varprojlim\Gamma/\Gamma(\pi^n)$ acts on the tower. View the inverse limit of cohomology as a module for $\mathbb{Z}_p[[G]] =: \Lambda$.

Theorem 4.7 (Lazard). Λ *is noetherian.*

Define

$$\tilde{H}^*(Y,\nu) := \varprojlim_m \varinjlim_k H^*(Y_k,\nu/p^m).$$

Theorem 4.8. *Suppose that* $\mathbb{G}$ *does not admit discrete series. Then*

- $\tilde{H}^*$ *are co-torsion* Λ*-modules*
- *Compare* $\mathbb{Q}_p\otimes\tilde{H}^{*G_K}$ *to* $H^*(Y_K,\nu)\otimes\mathbb{Q}_p$.