THE STEINBERG SYMBOL AND MODULAR SYMBOLS

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Inspiration/motivation: conjecture of R. Sharifi (and McCallum).

1. Sharifi's conjecture

The *Milnor K_n-group* associated to a commutative ring R is

$$K_n^M(R) := (R^\times \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} R^\times)/I$$

where I is generated by $a_1 \otimes \cdots \otimes a_n$ such that $a_i + a_j = 1$ for some $i \neq j$.

The *Steinberg symbol* is the map

$$\begin{aligned} R^\times \times \cdots \times R^\times &\to K_n^M(R) \\ (a_1, \ldots, a_n) &\mapsto a_1 \otimes \cdots \otimes a_n \bmod I =: \{a_1, \ldots, a_n\}. \end{aligned}$$

Suppose that (p, k) is an irregular pair; i.e., p divides B_k/k, where B_k is the k-th Bernoulli number. Let $R_n = \mathbb{Z}[\mu_p, 1/p]$. Let $E = R^\times/R^{\times p}$. Decompose $E = \bigoplus_{i=0}^{p-2} E^{(1-i)}$ where $E_{(1-i)}$ is the subgroup of E on which $G := \mathrm{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})$ acts via ω^{1-i}, where $\omega\colon G \to \mathbb{F}_p^\times$. We have $E^{(1)} \simeq \mu_p$, and $E^{(1-i)} = 0$ if i is even. Assume that i is odd from now on. Let ζ_i be a primitive p-th root of 1. We have $R^\times \to E^{(1-i)}$ sending $1 - \zeta_p$ to η_i. Let $\epsilon_i = \{\eta_{k-i}, \eta_i\} \in (K_2^M(R)/p)^{(2-k)}$.

Let f be a normalized cuspidal eigenform of level ℓ and weight k. Let $\mathfrak{p}$ be a prime above p. Suppose $f \equiv G_k \pmod{\mathfrak{p}}$. Then f gives rise to a modular symbol $\phi_f \in \mathrm{Symb}_{\mathrm{SL}_2(\mathbb{Z})}(\mathbb{C}[X, Y]_{k-2})$, namely

$$\phi_f((r) - (s)) = \int_s^r f(x)(zX + Y)^{k-2}\, dz$$

for $r, s \in \mathbb{P}^1(\mathbb{Q})$.

The universal L-value of ϕ_f is

$$\Lambda(\phi_f) := \phi_f((\infty) - (0)).$$

Define $L(\phi_f, i)$ by

$$\Lambda(\phi_f) = \sum_{i=0}^{k-2} \binom{k-2}{i} X^i Y^{k-2-i} L(\phi_f, i+1).$$

We have

$$\begin{aligned} \mathrm{Symb}_\Gamma(M) &\to H^1(\Gamma, M) \\ \phi &\mapsto (\gamma \mapsto \phi(\gamma(r) - (r))) \end{aligned}$$

for any $r \in \mathbb{P}^1(\mathbb{Q})$. Let ψ_f be the image of ϕ_f. We get an exact sequence defining the boundary symbols

$$0 \to \mathrm{Bound}_\Gamma(M) \to \mathrm{Symb}_\Gamma(M) \to H^1_{\mathrm{par}}(\Gamma, M) \to 0.$$

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Then

$$L(\psi_f, i+1) = L(\phi_f, i+1)$$

for $2 \leq i < k-2$.

Conjecture 1.1. Assume Vandiver's conjecture. Then there exist $\rho\colon (K_2^M(R)/p)^{(2-k)} \to \mathbb{F}_p(\omega^{2-k})$ and Ω^+ such that $\frac{L(\psi_f^+, i)}{\Omega^+} \equiv \rho(\epsilon_i)$ for all odd i.

We have

$$\rho\colon K_2^M(R)/p \to K_2(R)/p \simeq \mathrm{Cl}(\mathbb{Q}(\mu_p))/p \otimes \mu_p \to \mathbb{F}_p(\omega^{2-k}).$$

McCallum-Sharifi conjecture that if Vandiver's conjecture holds, then $K_2^M(R)/p \to K_2(R)/p$ is surjective.

Vandiver's conjecture comes in since

$$\dim_{\mathbb{F}_p} H^1_{\mathrm{Eis},\ell\neq p} = \dim_{\mathbb{F}_p} \left(\mathrm{Cl}(\mathbb{Q}(\mu_p))/p\right).$$

Theorem 1.2. *Assume $k < p$. There exists $\psi \in \left(H^1_{\mathrm{par}}(\mathrm{SL}_2(\mathbb{Z}), \mathbb{F}_p[X,Y]_{k-2})\right)^+$ such that given any ρ,*

(1) *$L(\psi, i) = \rho(\{\epsilon_i\})$ for i odd, $3 \leq i \leq k-3$, and*
(2) *$\psi|_{T_q} = (1+q^{k-1})\psi$ for $q = 2, 3$.*

We sketch the proof. Let $X_n = (\mathbb{Z}/p^n\mathbb{Z})^2 = \{(x,y) \in (\mathbb{Z}/p^n\mathbb{Z})^2 : (x,y,p) = 1\}$. A *Manin symbol* is a symbol on X_n with values in some module M.

Theorem 1.3. *There exists $e_n \in \mathrm{Manin}_{\Gamma_0(p^n)}(K_2(R_n)/2)$, and $e_n|_{T_q} - (q+\omega(q)) \in \mathrm{Bound}_{\Gamma_0(p^n)}(M)$.*

Let

$$(x,y) = \begin{cases} \{1-\zeta^x, 1-\zeta^y\}, & \text{if } xy \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

If $xy \neq 0$ and $x+y \neq 0$, then

$$\frac{\zeta^y(1-\zeta^x)}{1-\zeta^{x+y}} + \frac{1-\zeta^y}{1-\zeta^{x+y}} = 1.$$

Therefore $e(x,y) - e(x, x+y) - e(x+y, y) = 0$. (This is the 2-nd Manin relation.)

The action of T_2 is given as follows:

$$e_n|_{T_2}(x,y) = e_n(x, 2y) + e_n(2x, y) + e_n(x+y, 2y) + e_n(2x, x+y).$$

If $xy \neq 0$ and $x+y \neq 0$, then

$$\frac{(1-\zeta^{x+y})(1-\zeta^x)}{1-\zeta^{2x}} + \zeta^x \frac{(1-\zeta^{2y})(1-\zeta^x)}{(1-\zeta^{2x})(1-\zeta^{2y})} = 1.$$

Stevens generalized this by defining

$$\phi \in \mathrm{Symb}_{\mathrm{GL}_2(\mathbb{Q})}(\mathrm{Dist}(\mathbb{Q}^2, \mathcal{K}_n(R))).$$