Workshop on Nonholonomic Dynamics and Integrability (07w5029)

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Overview and introduction

Nonholonomic mechanics describes the motion of systems subbordinated to non-holonomic constraints, i.e., systems whose restrictions on velocities do not arise from the constraints on the configuration space. The best known examples of such systems are a sliding skate and a rolling ball, as well as their numerous generalizations. These systems usually exhibit very peculiar, often counter-intuitive, behavior. For example, a golf ball rolls inside a vertical tube, while oscillating and seemingly defying gravity, and a rattleback top (Celtic stone) spins only in one direction and resists spinning in the opposite one. Non-holonomic mechanics is also a cornerstone of the control theory, where the non-holonomic property is pivotal in the descriptions of attainable configurations; see [2, 5, 6, 9, 14].

The integrability vs. chaos dichotomy in such systems is one of their main points of interest, which is yet to be better understood. Furthermore, a more profound understanding of the relation between several competing paradigms in nonholonomic mechanics, the applications to control theory, as well as the similarities with Hamiltonian systems would be very important for further progress in the theory. This workshop served as a unique opportunity to bring specialists in these domains together and foster interactions between researchers with diverse and often complimentary backgrounds in nonholonomic mechanics and in the adjacent areas, including sub-Riemannian geometry, Hamiltonian systems, billiard theory, sub-elliptic operators, motion planning in robotics, and others.

Below we outline several major topics which emerged in many talks at the workshop.

1 Nonholonomic mechanics, symmetries, control theory and the Hamilton-Jacobi equation

Connections between nonholonomic mechanics and control were described in the talk by A. Bloch (University of Michigan). Systems subject to nonholonomic constraints have natural links to nonlinear control systems as the constraints often induce good controllability properties. He discussed the important distinction between kinematic and dynamic nonholonomic systems and described the different optimal control problems that arise for these two classes of systems. A. Bloch discussed integrable systems that arise naturally in optimal control of nonholonomic systems and in nonholonomic systems themselves, as well as aspects of stability and stabilization of such systems. He also showed how one can get asymptotic stability in certain classes of nonholonomic systems even in the absence of external dissipation.

Numerous examples show that momentum dynamics of nonholonomic systems is remarkably different from that of holonomic/Hamiltonian systems. For example, symmetries do not always lead to spatial momentum conservation as in the classical Noether theorem. D. Zenkov (North Carolina State University) in the talk *Momentum conservation, integrability, and applications to control* discussed various examples, including nonholonomic momentum conservation relative to the body frame and their role in the theory of integrable nonholonomic systems. A new integrable nonholonomic system was introduced in the talk, and other applications of momentum dynamics to control of nonholonomic systems were discussed.

However, the theory of the Hamilton-Jacobi equation becomes subtle in the nonholonomic context. This was the subject of the talk *What happened to the Hamilton-Jacobi equation* by L. Bates (University of Calgary). By looking at some examples he attempted to explain why there is no Hamilton-Jacobi equation in nonholonomic dynamics and the implications this has for the solvability of completely integrable nonholonomic systems, [4].

This was in a nice contrast with the talk *Hamilton-Jacobi theory for nonholonomic mechanical systems* by M. de Leon (CSIC Real Academia de Ciencias), who developed in his talk a Hamilton-Jacobi theory for nonholonomic mechanical systems. The results were applied to a large class of nonholonomic mechanical systems called Chaplygin systems, [13].

J. Sniatycki (University of Calgary) gave the talk *Conservation laws, symmetry and reduction.* For a non-holonomically constrained mechanical system he described the distributional Hamiltonian formulation of its dynamics, formulated a non-holonomic analogue of Noether's theorem, and discussed the notion of symmetry of such a system. He also discussed various types of constants of motion and singular reduction of symmetries, [18].

2 Rolling problems: geometry, symmetries, control

A. Agrachev (SISSA) gave a talk on *Rolling balls and octonions*, in which he discussed hidden symmetries of the classical nonholonomic kinematic system (a ball rolling over another ball without slipping or twisting) and explained the geometric meaning of basic invariants of the corresponding vector distributions, see [1].

R. Montgomery (UC Santa Cruz) continued this topic in the talk *G_2 and the rolling distribution* based on his joint paper [7] with Gil Bor. The act of rolling one surface along another surface without slipping or spinning defines a rank 2 distribution on 5-manifold, the 5-manifold being a circle bundle over the product of the two surfaces. This distribution is manifestly invariant under the product K of the isometry groups of the two surfaces. When the two surfaces are spheres then K is the product of two rotation groups, one for each sphere. However, something miraculous happens when the ratio of radii of the spheres is 1:3: the local symmetry group of the rolling distribution becomes much larger. This local automorphism group becomes the first exceptional Lie group, namely, the split real form of the Lie group G_2. A proof of this fact was described using explicit constructions and relying heavily on the theory of roots and weights for the Lie algebra of G_2.

The control theory for the problem of rolling appeared in the talk of V. Jurdjevic (University of Toronto) *Rolling sphere problems on spaces of constant curvature* (joint work with J. Zimmerman) [11]. The setting of the rolling sphere problem on Euclidean space $\mathbb{E}^n$ for $n \geq 2$ consists of determining the path of minimal length traced by the point of contact of the unit sphere $\mathbb{S}^n$ on $\mathbb{E}^n$ as it rolls without slipping between two specified points of $\mathbb{E}^n$ and from a given initial rotational configuration to a prescribed terminal rotational configuration [10].

In this lecture Jurdjevic presented the results, in which the rolling sphere problem is extended to situations in which a sphere $\mathbb{S}^n_\rho$ of radius ρ rolls on a stationary sphere $\mathbb{S}^n_\sigma$ of radius σ, and to the hyperbolic analogue in which the spheres $\mathbb{S}^n_\rho$, $\mathbb{S}^n_\sigma$ are replaced by the hyperboloids $\mathbb{H}^n_\rho$, $\mathbb{H}^n_\sigma$ having hyperbolic radii ρ, σ with $\sigma \neq \rho$. The notion of rolling is taken in an isometric sense; the length of the path of the point of contact is measured by the metric of the stationary manifold and the orientations of the rolling manifold are expressed by the elements of its isometry group. This larger geometric perspective, that encompasses both the Euclidean and the hyperbolic geometries, also includes the unit hyperboloid $\mathbb{H}^n$ rolling isometrically on $\mathbb{E}^n$.

3 Billiards and sub-Riemannian geometry

One of the most well known open problems in the theory of mathematical billiards is to prove that the set of periodic billiard trajectories has zero measure. This conjecture is motivated by spectral theory (Weyl asymptotics for the spectrum of the Laplace operator). Recently a new approach to this problem was developed using ideas from sub-Riemannian geometry and the theory of exterior differential systems [3]

It is known that a planar Birkhoff billiard cannot have a 2-parameter family of 3-periodic orbits (this fact was proved by a number of authors in different ways), while spherical billiard domains can (for example, the spherical geodesic triangle with right angles). Yu. Baryshnikov (Bell Labs) explained in the talk *Spherical billiards with many 3-periodic orbits* this phenomenon and described spherical billiards having this property: the boundary of such a billiard must contain three segments of the sides of an equilateral right triangle.

V. Zharnitsky (University of Illinois) in the talk *Periodic orbits in outer billiards* (joint work with A. Tumanov) described an adjustment of the exterior differential systems method to the study of periodic trajectories of outer billiards. He proved that if the set of 4-period orbits in the outer billiard has non-empty interior then the table has four corners that form a parallelogram [19].

4 Nonholonomic systems and Lie groups

Nonholonomic mechanical systems are not Hamiltonian. L. Garcia-Naranjo (University of Arizona) in the talk *Almost Poisson bracket for nonholonomic systems on Lie groups* described the dynamics of nonholonomic systems in term of a bracket of functions that fails to satisfy the Jacobi identity. Now one speaks of an almost Poisson bracket. This approach avoids dealing with Lagrange multipliers, but, in practice, is difficult to implement because it involves heavy computations in coordinates.

He considered the so-called LL and LR systems where the configuration space is a Lie group and both the Hamiltonian and the constraints have invariance properties. These invariance properties allowed him to give a geometric construction of a bracket for the description of the system on a reduced space. This construction avoids computations in coordinates and provides relatively simple formulas for the bracket. The idea involved in the construction generalizes the theories of Lie-Poisson and semidirect product reduction to the nonholonomic setting. The constraint functions of the resulting bracket are Casimirs, so the constraints are satisfied automatically.

Discretization of integrable nonholonomic systems on Lie groups was described by Yu. Fedorov (Universitat Politechnica de Catalunya). Recently the formalism of variational integrators (discrete Lagrangian systems) was extended to systems with nonholonomic constraints. Fedorov briefy described this formalism and applied it to the case when the configuration space is a Lie group G and the discrete Lagrangian is left-invariant, while discrete constraints are left- or right-invariant with respect to the action of G. As examples, he constructed discretizations of several classical integrable nonholonomic systems with an invariant measure, in particular, the celebrated Chaplygin nonholonomic sphere problem. It appears that the resulting discrete dynamics is similar to that of the continuous models. He also proposed a method of choosing left-invariant discrete nonholonomic constraints that ensures preservation of the energy integral in the discretizations. The conservation of an invariant measure in the discrete systems was also discussed, [8].

P. Lee (University of Toronto) gave a talk *Infinite-dimensional geometry of optimal mass transport* on nonholonomic distributions in the infinite-dimensional context (joint work with B. Khesin) [12]. He considered the following nonholonomic version of the classical Moser theorem: given a bracket generating distribution on a manifold, two volume forms of equal total volume can be isotoped by the flow of a vector field tangent to this distribution. These results were discussed in the talk from the point of view of an infinite-dimensional non-holonomic distribution on the diffeomorphism groups. Furthermore, in the 60's Arnold showed that the Euler equation can be thought of as the geodesic flow on the group of volume-preserving diffeomorphisms. In a similar fashion, Otto showed that the mass transport problem can be consider as the geodesic problem on the Wasserstein space of all volume forms with the same total volume. In particular, the Wasserstein space can be regarded as the quotient of the group of all diffeomophisms by the subgroup of volume preserving ones, while the geodesic flow on the diffeomorphism group, given by the Burgers equation, is closely related to that on the Wasserstein space. It turns out that this relation between diffeomorphism group and the Wasserstein space can be understood via Hamiltonian reduction.

5 Applications of control theory and sub-Riemannian geometry

Yu. Sachkov (Program Systems Institute) gave a talk *Maxwell strata and conjugate points in Euler's elastic problem*. In 1744 Leonard Euler considered the following problem on stationary configurations of elastic rod. Given a planar elastic rod with fixed endpoints and tangents at the endpoints, it is required to find possible profiles of the rod with the given boundary conditions. Euler derived differential equations for stationary configurations of the rod, reduced them to quadratures, and described their possible qualitative types. Such configurations are called Euler elastic.

The question on stability of Euler elastic was solved only in some partial cases. In the talk, a full solution to the problem of stability of Euler elastic was described. In addition to this local problem, the corresponding global optimal control problem was also considered. Stability of Euler elastic corresponds to local optimality of extremals of a certain optimal control problem. It is known that extremals cannot be optimal after Maxwell points (where distinct extremal curves with the same length and cost functional meet one another) or after conjugate points (at the envelope to the family of extremal trajectories).

The group of discrete symmetries of the system of extremals is generated by the group of discrete symmetries of the equation of a pendulum. Maxwell points are described via the study of fixed points of the action of this symmetry group. Maxwell points for all types of elastic are found, [17].

Another application to the problem of optimal laser-induced population transfer in n-level quantum systems was discussed by W. Respondek (INSA de Rouen) in the talk titled *Integrability and non-integrability of sub-Riemannian problems* (joint work with A. Maciejewski). This problem can be represented as a sub-Riemanniann problem on $SO(n)$ and it is known that for $n = 3$ the Hamiltonian system associated with PMP (Pontryagin Maximum Principle) is integrable. In the first part of the talk, he showed that this changes completely for n larger than 3. Namely, the adjoint equation of PMP does not possess any first integral independent of the Hamiltonian on the leaves of the symplectic foliation. In proving non-integrability he used the Morales-Ramis theory.

In the second part he showed that the above-mentioned integrability of the adjoint equation for $SO(3)$ is a particular case of a more general result. Namely, he proved that the adjoint geodesic equation for a 3-dimensional homogeneous sub-Riemannian space possesses an additional quadratic first integral if and only if the space is symmetric, [15].

Yet another application was *A simple example of the Arnold diffusion* outlined in the talk by M. Levi (Penn State University) on his joint work with V. Kaloshin. They gave a simple geometrical explanation of the Arnold diffusion. The idea was illustrated for the case of a particle in a periodic potential in $\mathbf{R}^3$, and, in a slightly different setting, for the geodesic flow with time-periodic metric.

6 Slipping and rolling toys, bicycles, and nonholonomic engineering problems

T. Tokieda (University of Cambridge) talked about *Slipping and rolling toys and their integrability.* He discussed, both on the board and through toy demonstrations, a number of nonholonomic problems which look integrable—conserved quantities, quasi-periodicity, etc.—but seem awkward to fit into the current models of integrability. They exhibit other peculiarities, such as chirality and finite-time singularity, and he argued that these ought to be a generic part of physically realistic models of nonholonomic integrability.

The engineering perspective on nonholonomic problems was presented by A. Ruina (Cornell University) in his talk *Some mechanics perspectives on non-holonomic constraints.* He showed (with video) that despite common mythology, equations of motion for non-holonomic systems can sometimes be found by simple means; that the most common non-holonomic systems, by virtue of their symmetry, cannot have the most interesting of non-holonomic features, the asymptotic stability. He argued that the word "non-holonomic" might sensibly be replaced with "skates and wheels". Furthermore, despite more than a century-long history, there are no established equations of motion for a reasonably-general non-holonomic bicycle (demonstrated with video), [16].

In addition to talks, numerous informal discussions took place. The overall volume of interaction between the participants was very high.

The bibliography below lists several books on non-holonomic systems, sub-Riemannian geometry and optimal control theory, as well as various papers related to the talks.

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