Banff International Research Station

for Mathematical Innovation and Discovery

Intuitive Geometry Workshop
and
Intuitive Geometry Day in Calgary

Abstracts

Geometric iterative processes

Gergely Ambrus
University College London and University of Szeged, Hungary

A typical geometric iterative process problem can be described as follows. Start with a (usually planar) point set, and in each step, add new points generated by some fixed procedure (e.g. add the incenter of any triangle determined by three points of the set). The result is a monotone increasing set of points, and we are interested in their union: depending on the initial set and the procedure, it can be a discrete or a dense set, or some more complicated structure. The talk contains a little survey of results of this type.

Recent constructions concerning some old packing and covering conjectures

András Bezdek
Auburn University and Alfréd Renyi Mathematical Institute, Hungary

According to a theorem of L. Fejes Tóth, the density of a covering of the plane by non-crossing congruent copies of a convex disc K is at least area $K/$area $K(6)$ where $K(6)$ denotes the maximum area of a hexagon contained in K. It is a long standing open question if the non-crossing assumption can be omitted from this result.

A convex disc is r-fat if it is contained in a unit circle C and contains a concentric circle c of radius r. Recently, A. Heppes showed that the above inequality holds without the non-crossing assumption if K is a 0.8561-fat ellipse. G. Fejes Tóth showed that the non-crossing assumption can be omitted if K is an r_0-fat convex disc with $r_0 = 0.933$ or an r_1-fat ellipse with $r_1 = 0.741$.

One natural approach to L. Fejes Tóth's original problem was to prove that any given covering of the plane with congruent copies of a convex disc can be modified to be crossing free without increasing its density. Recently W. Kuperberg constructed a covering by special congruent pentagons and conjectured that any covering with non-crossing congruent copies of that pentagon has larger density. The main part of this talk gives the proof of this conjecture and thus rules out the above mentioned approach to the problem of L. Fejes Tóth. In the remaining part of the talk other recent packing and covering constructions will be discussed.

On Tarski's plank problem for lattices

Károly Bezdek
Canada Research Chair, University of Calgary

In 1991 T. Hausel and the author jointly raised an analogue of Tarski's plank problem for lattices. The talk will survey the state of the art of that problem and of some related questions.

Paths with no small angle

Imre Bárány
Rényi Institute, Budapest, and University College London

Given a finite set, X, in the plane, does there exist a polygonal path p, whose vertex set is exactly X, such that all angles of p are at least one degree? This question, and its solution, will be discussed in this talk. The results are joint work with Attila Pór and Pavel Valtr.

Colourful Hadwiger Theorem

Javier Bracho
Instituto de Matemáticas, Universidad Nacional Autónoma de México

A colourful version of the classic Hadwiger's line transversal theorem is proved. Namely, if an ordered and 3-coloured (finite) family of compact convex sets in the plane is such that every three differently coloured sets have a transversal line compatible with the order, then there exists a line transversal to all the sets of one of the colours.

Circle-covering of the Hyperbolic Plane

Károly Böröczky
Eötvös University, Budapest

Given a triangle T in the hyperbolic plane, let us consider the three circular sectors of radius R centred at the vertices of T such that the angle of each sector coincides with the corresponding angle of T. Then the ratio of the sum of areas of the tree sectors to the area of T is commonly known as the associated density of the tree circles to T. Naturally, the sectors may not be contained in T. If T is the regular triangle of circumradius R then the associated density is denoted by $D(R)$. It was proved by L.Fejes Toth in 1952 that given a covering of the hyperbolic plane by circles of radius R, the associated density to any Delone triangle is at least $D(R)$.

Various examples constructed in the 1970's showed that it is not clear what should be the notion density of arrangements of circles in the full hyperbolic plane. These examples also showed that one should always specify the type of the cell decomposition, and consider the intersections of the circles with each cell. More precisely, we define the density of the arrangement in a cell to be the ratio of the sum of the area of the intersections of the cirles with the cell to the area of the cell. The talk discusses the following theorem: Given a covering of the hyperbolic plane by circles of radius R, the density of the arrangement in any Delone triangle is at least $D(R)$.

Covering by Convex Bodies

Gábor Fejes Tóth
Alfréd Rényi Mathematical Institute, Hungary

A classical theorem of Rogers states that for any convex body K in n-dimensional Euclidean space there exist a covering of the space by translates of K with density not exceeding $n \log n + n \log \log n + 5n$. Rogers' theorem does not say anything about the structure of such a covering. We show that for sufficiently large values of n there exists a covering by translates of K with density $cn \log n$ which is the union of $\log n$ translates of a lattice arrangement of K.

Some Things I Would Like to Know About the Triangle

Richard K. Guy
University of Calgary

Bremner has characterized those rational triangles which have rational Malfatti radii. The radical centres of the 32 triads of Malfatti circles form an interesting configuration, whose rationality has not yet been investigated. The 32 triads also have pairs of 'Descartes-Soddy' tangent circles. Are there any collinearities or concyclicities among their 64 centres? If so, are there any subsequent concurrencies? And are any of their 64 radii rational?

Line transversals in families of discs

Aladár Heppes
Rényi Institute, Budapest, Hungary

The talk gives an overview of old and new Helly type results concerning line transversals in families of congruent discs. Open problems will also be presented.

Isolated transversals

Andreas Holmsen
University of Bergen, Norway

A geometric transversal to a family of convex sets is called an *isolated transversal* if it corresponds to an isolated point in the space of transversals. For pairwise disjoint convex sets in the plane, if a line transversal is isolated then it is isolated by a subfamily consisting of at most *three* convex sets, an observation that can be used to show Hadwiger's transversal theorem. The observation also has is a nice generalization for *hyperplane transversals* in any dimension, made by Goodman and Pollack. Here we will discuss the case of isolated *line transversals* in dimension greater than 2.

Circumscribed Polygons of Small Area

Dan Ismailescu
Hofstra University

Given any strictly convex disk K and any positive integer $n \geq 3$, we prove that there exists a convex n-gon C_n, circumscribed about K and a convex $2n$-gon I_{2n}, inscribed in K such that

$$\frac{Area(I_{2n})}{Area(C_n)} \geq cos\,\frac{\pi}{n},$$

with equality when K is an ellipse. This generalizes a result of Chakerian who proved the above inequality for $n = 3$ and $n = 4$. As a consequence, for every positive integer $5 \leq n \leq 11$ we improve the best known bounds for $max\ min\ \frac{Area(C)}{Area(K)}$ where the maximum is taken over all convex disks K and the min

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Total curvature estimates for the shortest path on the boundary of an elongated convex body

Krystyna Kuperberg
Auburn University

The total curvature of a C^2 path $r(s)$ in $\mathbb{R}^n$ parameterized by arc length s is defined as the integral $\int_C |r''(s)|\,ds$. We give estimates for the total curvature of a shortest path connecting two points in the boundary of a convex body contained between two concentric circular cylinders.

Covering a plane convex body by its congruent negative homothetic copies

Zsolt Lángi
University of Calgary

Lassak and Vásárhelyi proposed the following problem. Let λ_k denote the smallest positive number such that any plane convex body C is covered by k translates of $-\lambda C$. Let us determine λ_k for small values of k. Presently, the exact values of onlyλ_1, λ_2 and λ_3 are known. In this talk, we discuss two estimates regarding λ_6 and λ_7. One of the estimates verifies a conjecture of Lassak and Januszewski.

Recent results in Minkowski geometry

Horst Martini
Chemnitz/Germany

In this talk we present various recent results in Minkowski geometry, i.e., in the geometry of finite-dimensional normed linear spaces. These results refer to applications of combinatorial geometry (e.g., in view of Jung's theorem in normed linear spaces), characterizations of special Minkowski spaces (such as inner product spaces, or Radon norms), certain classes of convex bodies in Minkowski spaces (e.g., reduced polytopes or bodies of constant width), and "Minkowskian analogues" of theorems from planar and spatial elementary geometry.

Recent Results on the Bezdek–Pach Conjecture

Márton Naszódi joint work with Zsolt Lángi
University of Alberta

The Bezdek–Pach Conjecture asserts that the maximum number of pairwise touching positive homothetic copies of a convex body in $\mathbb{R}^d$ is 2^d. This conjecture extends a theorem of Petty according to which the maximum number of pairwise touching translates of a convex body in $\mathbb{R}^d$ is 2^d. In 2006, the speaker proved that the quantity in question is not more than 2^{d+1}. Now, we present an improvement to this result in the case when the convex body is centrally symmetric, and prove the upper bound $\frac{3}{2} \cdot 2^d$.

Relationships between packing and covering in the lower dimensions

Ed Smith
Jacksonville State University

Some convex bodies, especially tiles, can both pack and cover the space in which they reside very efficiently. Other convex bodies, such as the ellipse, are rather poor at both. An instinctive expectation is that convex bodies which can be efficiently packed can be arranged in efficient coverings and that those which are poor at packing are poor at covering as well . This expectation, while generally correct, is also misleading to some extent. To better understand the relationship between packing and covering densities, it is helpful to examine ways in which coverings can be derived from packings. What we learn may be useful in regard to at least one well known open problem.

Convex Solids with Quadric Boundary

Valeriu Soltan
George Mason University

By a *convex solid* in the Euclidean space R^n we mean an n-dimensional closed convex set, distinct from the whole space and possibly unbounded. A *convex surface* in R^n is the boundary of a convex solid. This definition includes a hyperplane or a pair of parallel hyperplanes. We say that a convex surface in R^n is a *convex quadric surface* provided it is a connected component of a quadric surface, given by the equation

$$\sum_{i,k=1}^{n} \alpha_{ik} x_i x_k + 2 \sum_{i=1}^{n} \beta_i x_i + \gamma = 0.$$

The talk deals with the following characterization of convex quadric surfaces.

Theorem. Given a convex solid $B \subset R^n$, $n \geq 3$, and a point $p \in B$, the following conditions are equivalent:

1) the boundary of B is a convex quadric surface or B is a convex cone with apex p,

2) all sections of the boundary of B by 2-dimensional planes through p are convex quadric curves.

Arrangements of the log curve

József Sólymosi
University of British Columbia

There are planar curves with the following property: Any arrangement of the translates of the curve is combinatorially equivalent to an arrangement of lines. We are looking for diffeomorphisms of the plane that map the translates of a curve into the set of lines. Such curves have various applications to discrete geometry, Ramsey theory, and number theory.

Diameters in 3-space

Konrad Swanepoel
University of South Africa

The *diameter graph* of a set of n points in R^3 of diameter D is obtained by joining pairs of points at distance D. This talk looks at some old and new properties of these graphs.

Packing three spheres into a minimal convex polytope of given shape

István Talata
Szent István University, Budapest

Let P be a given convex polytope in $\mathbf{R}^d$ $(d \geq 2)$, and let $r_1, r_2, r_3 > 0$ be given constants. If $c > 0$, then we use the notation $cP = \{cx \in \mathbf{R}^d \mid x \in P\}$. cP is a polytope that is a homothetic copy of P with coefficient of homothety c. We provide an algorithm to determine the smallest coefficient of homothety $\lambda > 0$ such that three spheres of radii r_1, r_2 and r_3, respectively, can be packed (without overlapping) into the polytope λP.

Stabbing numbers of convex and orthogonal subdivisions

Csaba Tóth
University of Calgary

It was shown by Chazelle, Edelsbrunner, and Guibas that for every subdivision of the plane into n convex cells, there is a line that stabs $\Omega(\log n/\log\log n)$ cells, and this bound is sharp, which means that the plane can be subdivided into n convex cells such that every line stabs $O(\log n/\log\log n)$ cells. No nontrivial bound is known for higher dimensional convex subdivisions.

Better bounds are possible for box subdivisions and axis-parallel lines: For every subdivision of the d-dimensional Euclidean space, $d \geq 2$, into n axis-aligned boxes, there is an axis-parallel line that stabs at least $\Omega(\log^{\frac{1}{d-1}} n)$ boxes. This bound cannot be improved. Furthermore, for every k and d, with $1 \leq k < d$, there is an axis-aligned k-flat that stabs at least $\Omega(\log^{1/\lfloor (d-1)/k \rfloor} n)$ boxes of the subdivision. We conjecture that this lower bound is sharp for every $1 \leq k < d$.

Typical faces of best approximating polytopes with a restricted number of edges

Viktor Vígh
University of Szeged, Hungary

We consider best approximating polytopes with a restricted number of edges of convex bodies with a C^2 boundary. The goodness of approximation is measured by Hausdorff distance. The asymptotic formulae for the distance of the convex body K and the best approximating inscribed P_n^i and circumscribed P_n^c polytopes with at most n edges are

$$\delta_H(K, P_n^i),\ \delta_H(K, P_n^c) \sim \frac{1}{2n} \int_{\partial K} \kappa^{1/2}(x) dx \quad \text{as } n \to \infty.$$

In this talk, we discuss the shape of the typical faces of the above best approximating polytopes. We show that the typical faces are squares in a suitable sense.

Map operations and k-orbit maps

I. Hubard, A. Orbanic and **A.I. Weiss**
York University

A map with exactly k flag-orbits under the action of its automorphism group is called a k-orbit map. We determine the classes of k-orbit maps for $k \leq 4$ and describe the tools to determine the classes in general. We also investigate medials and truncations of such maps.

Characterizing sets in E^n which are rationally realizable in some E^m

Joseph Zaks
University of Haifa, Israel

A set B of real numbers in the line E^1 is called simply-irrational if all the numbers of B are rational multiples of a single real number r for which r^2 is a rational number, i.e., B is a subset of the affine image rQ of the rational line Q.

A set of points C in E^n is called rationally realizable if, for some m, there exists a set D in E^m, which is congruent to C, such that all the points of D are rational points.

The following has been shown in [1].

Theorem 1: A set S of points in E^n is rationally realizable if, and only if, all the mutual distances between pairs of its points have rational squares; moreover, if d is the dimension of the affine hull of S, then S can be rationally realizable in E^{4d}.

The purpose of this note is to extend this result, in showing the following.

Theorem 2: If a set S of points in E^n has the property that all the mutual distances between pairs of points of S have rational squares, and if the affine hull of S is of dimension d, then S is congruent to a subset T of E^d , which is the Cartesian product of d simply-irrational linear sets, i.e., T given by $T = \{(x_1, \ldots, x_d) \in E^d \mid x_i$ is a rational multiple of r_i for all $i, 1 \leq i \leq d\}$, where $r_1, \ldots, r_d$ are some d positive real numbers, each one of which has a rational square.

The following is an interesting consequence of Theorem 2.

Theorem 3: If a set S of points in E^n for which all the mutual distances between pairs of points of S have rational squares, then the set S has the property that for every k, if H_k is an affine flat of dimension k in E^n, then all the k-dimensional volumes of k-simplices, determined by $k+1$ points of S in H_k, form a simply-irrational set, i.e., a subset of the affine image rQ of the rational line Q, for some real number r for which r^2 is a rational number; moreover, parallel affine k-flats have the same r.

References

[1] Joseph Zaks, *The rational analogue of the Beckman-Quarles Theorem and the rational realization of some sets in E^d*, Rend. Mat. Appl. (7) **26** (2006), 87–94.
