MULTI-AGENT OPTIMIZATION (5)

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Multi-Agent Optimization

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- 1. Variational Analysis Tools
- 2. Deterministic Problems
  - foundations & computational schemes
- 3. Stochastic Problems (Walras)
  - foundations & computational schemes

III. Stochastic Models

Outline

1. Incomplete markets

2. Equilibirum for incomplete markets

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Classical Arrow-Debreu Model

- $\mathcal{E}$ = exchange of goods $\in \mathbb{R}^n$
- (economic) agents: $i \in \mathcal{I}$, $|\mathcal{I}|$ finite
  consumption by agent i: $x_i \in \mathbb{R}^n$
  endowment: $e_i \in \mathbb{R}^n$, utility: $u_i : \mathbb{R}^n \to [-\infty, \infty)$,
  survival set: $X_i = \operatorname{dom} u_i = \left\{ x_i \,\middle|\, u_i(x_i) > -\infty \right\}$
- exchange at market prices: p
- i-budgetary constraint: $\langle p, x_i \rangle \leq \langle p, e_i \rangle$

The agents: $i \in \mathcal{I}$, $|\mathcal{I}|$ finite

- information: present state & all potential future states $s \in S$
- beliefs: agent-i assigns 'probability' $b_i(s)$ to (future) state s
- activities: y_i $[= y_i^0]$, input/output: $T_i^0 y_i \to T_i^1(s) y_i$
- securities ($\approx$ future contracts) z_i $[= z_i^0]$
- consumption: $(x_i^0, (x_i^1(s), s \in S))$

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The agents: $i \in \mathcal{I}$, $|\mathcal{I}|$ finite

- criterion: max $u_i^0(x_i^0) + E_i\{u_i^1(\boldsymbol{s}, x_i^1(\boldsymbol{s}))\}$
  = max $u_i^0(x_i^0) + \sum_{s\in S} b_i(s) u_i^1(s, x_i^1(s))$,
  more generally: $U_i(x_i^0, (x_i^1(s), s \in S))$
- survival set (feasible consumption): $X_i = \mathrm{dom}\, U_i$
  $= \{x_i^0, (x_i^1(s), s \in S)) \mid U_i(x_i^0, (x_i^1(s), s \in S)) > -\infty\}$
- U_i usc, concave, ‘increasing’, insatiable (in all states)
- $\Longrightarrow X_i$ convex, $\not\Longrightarrow X_i$ closed
- ‘increasing’ $\Longrightarrow X_i + [\mathbb{R}^n_+ \times (\mathbb{R}^n_+)^S] \subset X_i$, $\mathrm{int}\, X_i \neq \emptyset$,
- endoments. primary goods $e_i^0, (e_i^1(s), s \in S)$
  secondary goods (typically shares) $\tilde{e}_i^0$
- primary goods: tradable and fixed supply, in all states
- secondary goods: tradable, no consumption, $\sum_{i\in\mathcal{I}} \tilde{e}_i^0 > 0$

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Market prices for goods

- $(p^0 \neq 0, (p^1(s) \neq 0, s \in S))$ for primary goods
- *numéraire prices* w.r.t. $g \in \mathbb{R}^n_+$ when
  $$\langle p^0, g \rangle = 1, \quad \forall\, s, \langle p^1(s), g \rangle = 1$$
- $\tilde{p}^0$ for trading of secondary goods (possibly = 0)

Activities

Transform goods at $t = 0$ into goods at $t = 1$, $T_i^{\cdot}$ technologies
Activities available to agent i: $j = 1, \ldots, J_i$ (also $J_i = 0$)
Input: $T_i^0 y_i \in \mathbb{R}^n_+$, $\tilde{T}_i^0 y_i \in \mathbb{R}^n_+$
Output: $T_i^1(s) y_i \in \mathbb{R}^n_+$
Assumption: input required and output produced > 0 for all j

Examples:

- savings as an activity
- bond and stock holding (cash flow)
- production: home production,
- firms: primary and secondary inputs; profit-focused firm

Securities

Securities: finitely many types (of contracts) $k = 1, \dots K$
unlimited in quantity
with delivery in primary goods ≥ 0 at $t = 1$ at prices $p^1(s)$.
1 unit of contract k requires delivery $D_k(s, p^1(s))$ at $t = 1$
$D(s, p^1(s))$ delivery matrix
Additional assumptions:

- $\exists\, s \in S: \;\; D_k(s, p^1(s)) \neq 0$ for all $p^1(s) \neq 0$
- $p^1(s) \mapsto D(s, p^1(s))$ continuous,
- insensitive to price scaling:

$$D(s, \lambda p^1(s)) = D(s, p^1(s)), \; \lambda > 0$$

Examples: Financial instruments, Derivatives, Futures, etc.

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A derivative as a security

Example

$\tilde{D}_k(s, p^1(s)) = \beta_k \max[0, p^1_l(s) - K_l]$, call option
satifies continuity w.r.t. $p^1(s)$, but not nontriviality, price scaling

with numéraire,

$$D_k(s, p^1(s)) = \frac{\beta_k}{p^1_{num}(s)} \Big(\max[0, p^1_l(s) - p^1_{num}(s) K_l] + \theta\, p^1_{num}(s) \Big)$$

θ: transaction fee

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Securities market

Long and short position:

z_i^+ purchases, z_i^- sales of agent-i

deliveries: $D(s, p_1(s))[\, z_i^+ - z_i^- \,]$

Purchase price of security k: $q_k \geq 0$ at $t = 0$.

$\langle q, [\, z_i^+ - z_i^- \,]\rangle$ net amount 'paid' by agent-i

implicit: 'Broker entity' (versus 'individual' contracts)

Completeness: complete if $D(s, p^1(s))$ of full rank, i.e.,

every vector in $\mathbb{R}^n_+$ achied as $D(s, p^1(s))z$.

Incompleteness: no such assumption, only nontriviality

Excluding defaults: Premium prices

Promises for delivery can't exceeds availabilities
Premium to be charged when supply gets tight for deliveries
Bonus for contribution of goods rather than consumption
$\Longrightarrow$ Double Market

top-priority market in obtaining deliveries
at prices $p^{1+}(s) \in \mathbb{R}^n_+$
premiums: $0 \le r(s) = p^{1+}(s) - p^1(s)$
$r_l(s)$ 'ensures' availability of good l in state s

agent-$\hat{\imath}$ long on k pays $\langle p^{1+}(s), D_k(s, p_1(s))\rangle$
agent-i with $e_i^1(s)$ and $T_i^1(s)y_i$ gets paid at price $p^{1+}(s)$.

Price system: $p^0, \tilde{p}^0, q, [\,(p^1(s), p^{1+}(s)), s \in S\,]$

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Agent's optimization problem

Given a price system:

$$
\begin{aligned}
&\max\ U_i(x_i^0, x_i^1(\cdot)) \quad \text{so that}\\
&\langle p^0, x_i^0 + T_i^0 y_i\rangle + \langle \tilde{p}^0, \tilde{T}_i^0 y_i\rangle + \langle q, z_i^+\rangle\\
&\qquad\qquad \le \langle p^0, e_i^0\rangle + \langle \tilde{p}^0, \tilde{e}_i^0\rangle + \langle q, z_i^-\rangle\\
&\langle p^1(s), x_i^1(s)\rangle + \langle p^{1+}(s), D(s, p^1(s)) z_i^-\rangle\\
&\quad \le \langle p^{1+}(s), e_i^1(s) + T_i^1(s) y_i\rangle + \langle p^1(s), D(s, p^1(s)) z_i^+\rangle,\ \forall\, s\\
&\qquad (x_i^0, x_i^1(\cdot)) \in X_i,\ \ y_i \ge 0,\ \ z_i^+ \ge 0,\ \ z_i^- \ge 0
\end{aligned}
$$

Free disposal $\Longrightarrow\ \le$ in the constraints

Equilibirum definition

- prices $(\bar{p}^0, \tilde{p}^0, \bar{q}, [\,(\bar{p}^1(s), \bar{p}^{1+}(s)), s \in S\,])$ such that
- $(\bar{x}_i^0, \bar{x}_i^1(\cdot), \bar{y}_i, \bar{z}_i^+, \bar{z}_i^-)$ optimal for agent-i

and market clearing

- $\sum_i (\bar{x}_i^0 + T_i^0 \bar{y}_i - e_i^0) \leq 0$ with $=_l$ if $\bar{p}_l^0 > 0$
- $\sum_i \tilde{T}_i^0 \bar{y}_i - \tilde{e}_i^0) \leq 0$ with $=_l$ if $\tilde{p}_l^0 > 0$
- $\sum_i \bar{z}_i^+ \leq \sum_i \bar{z}_i^-$ with $=_k$ if $\bar{q}_k > 0$
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- $D(s, \bar{p}^1(s) \sum_i \bar{z}_i^- \leq \sum_i (e_i^1(s) + T_i^1(s)\bar{y}_i)$
  with $=_l$ if $\bar{p}_l^{1+}(s) > \bar{p}_l^1(s)$

last condition: handles 'collaterals, default penalties, ...'

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Existence

Theorem

Under strict survivability, the existence is assured; $\exists$ is an equilibrium with $\sum_{i=1}^{I} \bar{z}_i^+ = \sum_{i=1}^{I} \bar{z}_i^-$.

Strict survivability. For each agent i there is a choice of $\hat{x}_i^0$ and $\hat{x}_i^1(\cdot)$ satisfying $(\hat{x}_i^0, \hat{x}_i^1(s)) \in X_i(s)$ and $\hat{y}_i \geq 0$ such that

$$\text{for primary goods:} \quad \begin{cases} \hat{x}_i^0 + T_i^0 \hat{y}_i < e_i^0, \\ \hat{x}_i^1(s) < e_i^1(s) + T_i^1(s)\hat{y}_i \quad \text{for } s \in S, \end{cases}$$

$$\text{for secondary goods:} \quad \tilde{T}_i^0 \hat{y}_i \leq \tilde{e}_i^0$$

Proof. via optimality analysis of agent's prolbem. □

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Proof. via optimality analysis of agent's prolbem. $\square$

Optimality conditions: N. & S.

Under strict survivability, $(x_i^0, x_i^1(\cdot), y_i, z_i^+, z_i^-)$ is optimal if feasible, $\exists$ probabilities $\pi_i(s) > 0$, factors $\mu_i > 0$, $\rho_i > 0$:

- $(x_i^0, x_i^1(\cdot))$ maximizes over X_i,

$$\mu_i U_i(x_i^0, x_i^1(\cdot)) - \langle p_0, x_i^0 \rangle - \rho_i \sum_{s \in S} \pi_i(s) \langle p_1(s), x_i^1(s) \rangle$$

- for each activity $j = 1, \ldots, J_i$,

$$\langle p_0, T_{i,j}^0 \rangle + \langle \tilde{p}_0, \tilde{T}_{i,j}^0 \rangle \geq \rho_i \sum_{s \in S} \pi_i(s) \langle p^{1+}(s), T_{i1,j}(s) \rangle, \;\; y_{i,j} = 0 \text{ if } \; >$$

- for each asset $k = 1, \ldots, K$,

$$q_k \geq \rho_i \sum_{s \in S} \pi_i(s) \langle p^1(s), D_k(s, p_1(s)) \rangle, \;\; z_{i,k}^+ = 0 \text{ if } \; >$$

- for each asset $k = 1, \ldots, K$,

$$q_k \leq \rho_i \sum_{s \in S} \pi_i(s) \langle p^{1+}(s), D_k(s, p_1(s)) \rangle, \;\; z_{i,k}^- = 0 \text{ if } \; <$$

- budget constraints are ‘satisfied’ as equations.

Subjective probabilities, discount rates

Definition

- μ_i converts utility into the scale of prices at time 0 and is the *utility price* for agent i.
- $\pi_i(s)$: like *risk-neutral probabilities* of state s revealed for agent i in response to the given price system.
- ρ_i can be viewed as the *discount rate* of agent i for converting prices at time 1 into prices at time 0 (not necessarily ≤ 1); more appropriate in the case of a numéraire price system ($\Rightarrow \leq 1$).

Imputed values: activities & securities

Discount rate ρ_i relative to $(p_0, p_0', q, p_1(\cdot), p_1^+(\cdot))$, is so that, w.r.t. the 'probabilities' $\pi_i(s)$,

$$\langle p^0, T_i^0 y_i\rangle \geq \rho_i \sum_{s\in S} \pi_i(s)\langle p^{1+}(s), T_i^1(s) y_i\rangle \;\; \forall\, y_i$$

$$\langle q, z_i^+\rangle \geq \rho_i \sum_{s\in S} \pi_i(s)\langle p^1(s), D(s, p^1(s)) z_i^+\rangle \;\; \forall\, z_i^+$$

$$\langle q, z_i^-\rangle \leq \rho_i \sum_{s\in S} \pi_i(s)\langle p^{1+}, D(s, p^1(s)) z_i^-\rangle \;\; \forall\, z_i^-$$

Hold as equations when (y_i, z_i^+, z_i^-) are part of a solution to agent-i's problem.

Further . . .

- redundant security positions
- absence of arbitrage: (z_i^+, z_i^-)
- Variational representation: global versus disaggregated