Conformal and CR geometry: Spectral and nonlocal aspects

A. Rod Gover (University of Auckland),
Andreas Cap (University of Vienna)

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1 Overview of the Field

Conformal geometry has occupied an important position in mathematics and physics since early last century when Bateman, Dirac and others pointed out the conformal invariance of the fundamental equations governing electromagnetism and massless spin particles. More recently it has turned out to play an important role in many areas of mathematics and physics. Often these connections are rather subtle and unexpected. For example the smooth boundary of a domain in complex n-space inherits a so-called CR structure. As discovered by Ch. Fefferman, a CR structure is equivalent to a special conformal geometry. Thus there is an intimate relationship between conformal geometry and complex analysis. In the setting of elliptic PDEs and hard geometric analysis, certain core problems based on Riemannian geometry have yielded to conformal approaches. The touchstone in this area is the Yamabe problem of rescaling a metric to one with constant scalar curvature; this was finished by Schoen for structures of positive definite signature on compact manifolds in 1984, building on major earlier contributions of Aubin (1976), Trudinger (1968), and Yamabe (1960). The Yamabe problem is a semilinear PDE with a critical power nonlinearity. This may be viewed as a "nonlinear variant" of a certain conformally invariant linear equation. Many of the recent breakthroughs in Physics have involved conformal geometry in some essential way; the latest examples being string theory, and the AdS/CFT (anti-de Sitter/conformal field theory) correspondence.

The basic setup for conformal geometry is a (suitably) smooth manifold equipped with a conformal structure, that is an equivalence class of Riemannian metrics; two metrics are equivalent if they are related by multiplying by a positive smooth function. Thus, for example, on a conformal manifold one can measure angles between tangent vectors, but not the length of a tangent vector. This leads to immediate difficulties. Weyl's invariant theory classifies all Riemannian (or pseudo-Riemannian) invariants and invariant differential operators between natural bundles. However the corresponding problem for conformal geometry is much more difficult and is still not completely solved. The essential difference between these problems can be seen in the model spaces for the two geometries (Riemannian and conformal).

Each of the model spaces is a homogeneous space G/H. In the Riemannian case, H is the orthogonal group, so its representation theory is very well understood. In the conformal case the stabiliser H of a point is a parabolic subgroup of G=SO(p+1,q+1) (where (p,q) is the signature of the conformal structure). The finite-dimensional representations of H are not completely reducible in general and there is no complete classification of representations.

On the one hand the "parabolic" nature of conformal geometry certainly slowed its early development, but, on the other hand, it has encouraged a strong two-way interaction between the study of conformal structures and deep aspects of representation theory. For example conformally invariant operators (differential and pseudo-differential) give intertwinors between appropriate representation spaces. On the other hand the algebraic ideas of Kostant, Zuckermann and others have played a key role in the recent rapid progress in the construction and understanding of conformally invariant differential operators (as in for example Eastwood's "curved translation principle", pioneered in [12], and the "Bernstein-Gelfand-Gelfand [BGG] machinery" of Cap, Slovák, and Souček [10]).

On the geometric side a breakthrough in the modern theory was Fefferman's article "Parabolic invariant theory in complex analysis," [13]. This was concerned with CR geometry. The parallel development for conformal geometry (due to Fefferman and Graham and known as the "Fefferman-Graham ambient metric construction") came soon after, in 1985. In fact the article released at that point was a conference announcement which sketched the main ideas and potential applications. The detailed version of the article has been publicly archived in October of this year, see [14]; of course this necessarily links in the huge volume of work over the last two decades based around these constructions. The original works of Fefferman and Graham were motivated by invariant theory but established powerful geometric constructions that can be applied effectively to many problems related to these structures. A common theme is to use the constructions to turn geometric problems into well defined algebraic problems in parabolic representation theory. For the invariant theory Bailey, Eastwood, and Graham solved many of the key problems for the conformal and CR cases, in [2] . This was partly based on ideas from the analogous projective problem as solved by Gover in [16].

Linking several of the recent developments is a tool known as tractor calculus. For example this is the main geometric tool underlying the curved translation principle and the BGG machinery mentioned above. In the conformal setting the basic structures of tractor calculus were given by T.Y. Thomas and E. Cartan early-mid last century. (Indeed the basic tractor bundle and connection may be viewed as an induced structure for the Cartan principal bundle and connection.) However the calculus has been developed significantly recently by ourselves and our collaborators (especially Bailey, Eastwood and Graham). Through this, for example, we now have a direct link between the ambient constructions of Fefferman and Graham and the normal Cartan conformal and CR connections. Since the Cartan/tractor machinery fits well with representation theory, while the ambient metrics allow for considerable geometric simplification this link is a powerful tool which is being increasingly exploited.

On the geometric analysis side, an intriguing thread over the past decade or so is the problem of prescribing the Q-curvature. This is an invariant (due to Thomas Branson) which acts very much like the Gauss curvature in dimension 2 – the PDE describing its prescription is semilinear, and has an inhomogeneity and an exponential nonlinearity. For example, recent work of Djadli and Malchiodi shows that there is a Q-charge (basically the integral of Q) which, under pinching-type conformal changes that change the topology, "bubbles off" in a quantised way, much like the Yang-Mills topological charge. Furthermore, they prove existence of a metric with constant Q–curvature in the case where the charge is not an integral multiple of the bubbling charge. In other recent work, Graham and Hirachi have shown that the integral of Q has, as its total metric variation, the Fefferman-Graham obstruction tensor – the even dimensional obstruction to extending the Fefferman-Graham ambient metric construction to all orders. As such, it is an important object for the AdS/CFT correspondence. Recent work of Graham and Zworksi [21], Fefferman and Graham [14], Chang, Qing, and Yang [preprint], and Albin [1] makes the connection between the Q- curvature and AdS/CFT. At the same time, the Chang-Qing-Yang paper links this up with developments of about a decade ago, when it was established that a functional based on Q gives the main term in higher-dimensional Polyakov formulas, i.e. formulas for the quotient of functional determinants (of, for example, the Yamabe operator) at conformally related metrics. That is, this work provides a link to the role played by the Q-curvature in string theory.

2 Open Problems and Recent Developments

In recent work the applicants and collaborators have made several closely linked discoveries. These include: invariant operator non-existence results, [18], the existence of a new classes of conformally invariant elliptic differential complexes on conformally curved structures [3], new torsion quantities for these "detour complexes" which generalise Cheeger's 1/2-torsion [articles in progress]; subcomplexes in Bernstein–Gelfand–Gelfand sequences [8]; operators between differential forms which generalise the Q-curvature. An efficient version of tractor calculus to exploit the relation between conformal and CR geometry introduced by Fefferman [7].

The original aim of the meeting was to exploit these tools to construct and study global invariants for the global analysis and spectral theory on manifolds, however we were drawn in another direction due to an unexpected new development. Elliptic differential complexes and their generalisations provide a key tool for obtaining global invariants. For example on the locally flat structures there are the BGG complexes of [10]; these capture integrability conditions and their cohomologies give global invariants. However in the curved setting finding invariant differential sequences that yield useful complexes is non-trivial and the problem is for the most part completely open; some progress and interesting new directions have been mentioned already, the detour complexes of e.g. [3, 4, 19] and the subcomplexes of [8], which lead to deformation complexes for various parabolic geometries, see [5].

Investigations into the construction and application of invariant operators is often frustrated by the complexity of the operators involved and their constructions. Thus any steps toward a simplification and unification of the picture presents an important development. Very recently a new discovery was made which offers the possibility of a vast simplification in the construction and understanding of invariant operators and so our activity focussed on developing this new tool and and its applications. The central tool is the so-called *curved Casimir operator* of [9]. For G a semi-simple Lie group one has the quadratic Casimir that acts on induced G-representations. This is very well understood in the case of representations induced from a maximal compact subgroup, in which the Casimir essentially is a Laplacian. In [9] the authors show that in the parabolic case, the Casimir is an operator of order at most one, which admits a natural generalisation to to an invariant differential operator on any curved parabolic geometry. A parabolic geometry of type (G, P) is a manifold equipped with a P-principal bundle $\mathcal{G}$ (which via a soldering form type relation may be viewed as a generalisation of the idea of a higher order frame bundle) and on this a canonical Cartan connection which gives a suitably equivariant total parallelism, in a way that generalises the parallelism of G by the Maurer-Cartan form. By classical results of Cartan, Tanaka, and Chern–Moser, conformal and CR structures admit an equivalent description as parabolic geometries. It was shown in [6] that the exterior derivative on the total space of $\mathcal{G}$ descends to an invariant first order operator D, known as the fundamental derivative. This first order operator maps sections of an arbitrary natural vector bundle to sections of the tensor product of that bundle with the so–called adjoint tractor bundle. In terms of the fundamental derivative, the curved Casimir $\mathcal{C}$, acting on an arbitrary vector bundle associate to $\mathcal{G}$ is $(B \otimes id) \circ D^2$. Here B is the metric on the adjoint tractor bundle arising from the Killing form on $\mathrm{Lie}(G)$. Hence the first order operator $\mathcal{C}$ is a Laplacian-like contraction of the second power of D. It was shown in [9] that the naturality properties of the fundamental derivative carry over to $\mathcal{C}$.

3 Scientific Progress Made

The article [9] contains a systematic construction for a class of differential operators via curved Casimirs. This construction produces splitting operators, which form an essential ingredient in all constructions related to the curved translation principle and BGG sequences as described above. During the meeting we have extended these ideas to provide a systematic way for a direct (i.e. without going through iterated translations) construction of invariant linear differential operators acting between bundles associated to irreducible representation. This method also leads to non–standard operators and works in singular infinitesimal character, and hence produces operators which cannot be constructed using the BGG machinery. We have also developed a general machinery to compute the symbols of the resulting operators and hence in particular prove that they are non–zero (which is a major difficulty in the theory).

These basic tools work for arbitrary parabolic geometries. To check the power of our results we have developed some applications in conformal geometry; For this structure there are many existing results to compare to. Apart from several explicit examples, we have developed a general construction which produces almost all the formally self-adjoint operators between irreducible tensor bundles. (These are exactly the operators one does no obtain via the standard BGG machinery of [10, 11]). For each appropriately weighted irreducible tensor bundle E the construction gives for every dimension $n \geq 3$ a differential operator $P : E[k - n/2] \to E[-k - n/2]$ and we prove that, for all but a finite set of dimensions the operator is of order $2k$ with non–zero symbol of that order (and in particular non-trivial). This family includes, for example, operators with leading term a power of the Laplacian acting on densities and so, as one of the simplest special cases, we obtain a new construction the conformal powers of the Laplacian along the lines of the Graham-Jenne-Mason-Sparling operators [20].

The striking feature of the construction is its simplicity of the construction and its uniformity. For all parabolic geometries these operators are given explicitly by polynomials of the form $L = (\mathcal{C} - \lambda_0)(\mathcal{C} - \lambda_1)\cdots(\mathcal{C} - \lambda_\ell)$ applied to the section space $\mathcal{V}$ of natural bundles V. The constants λ_i that have to be used in each case are Casimir eigenvalues of irreducible subquotients of V, and hence can be computed explicitly using finite dimensional representation theory. If there are coincidences between these eigenvalues (which can be forced by appropriately adapting the conformal weight) then L determines an operator between the corresponding pairs of irreducibles. (If all these eigenvalues are different, one obtains the splitting operators constructed already in [9]).

The more difficult side of the construction is to establish information on the operators constructed. An obvious minimal goal here is to calculate the symbol; in fact a knowledge of the symbol is also sufficient for many applications. Calculating this is complicated by the fact that starting with a representation inducing V, the number of factors in the composition defining L is in general much higher than the order of the resulting invariant operator. So although the symbol of a single curved Casimir well understood, this does not easily lead to information on the symbol of L. To compute the latter, we developed an algorithm to rewrite the (manifestly invariant) operator L as a composition of the order-of-L number of first order operators; each of the first order operators involved is not individually invariant but once one overall and simple choice is made then these are determined systematically by the operator and the algorithm. The symbols of these first order ingredients can be computed using purely algebraic tools, hence reducing the computation of the symbol to a combinatorial problem. Proving that the symbol of the operator L is nonzero in all but finitely many dimensions uses the information from the algorithm but also draws significantly on results from finite dimensional representation theory. Remarkably, both arguments based on compact groups, and arguments based on weights (and hence on complex groups and Lie algebras) are used to establish the result.

Given all these features it seems the constructions we developed during the RIT session, combined with the earlier observations in [9], will have a significant international impact on the approach researchers use to construct invariant operators and related tools such as differential complexes.

It is difficult to imagine that these developments could have been achieved without an ideal setting such as that provided by BIRS and the RIT programme. We were working intensively during the whole period of the programme, switching between the development of general ideas and computing explicit examples; the latter were often were quite involved. Without being able to completely concentrate on this topic for an extended period, similar progress would certainly not have been possible.

4 Outcome of the Meeting, a Summary

We feel the RIT meeting was tremendously successful. We developed a simple approach to constructing invariant differential operators which applies uniformly to the entire class of parabolic geometries. This approach was tested for conformal structures, where it leads to very strong results and also proves its computational efficacy. The general theory surrounding the curved Casimir we developed, and the specific results obtained, demonstrate that the curved Casimir plays a fundamental role in the invariant operator theory of parabolic geometries. For some directions it seems likely that this will lead to a paradigm shift in the approach to constructing and treating invariant operators. Based on the work carried out at the BIRS meeting we have two articles in progress, with one near completion.

References

[1] P. Albin, A renormalized index theorem for some complete asymptotically regular metrics: the Gauss-Bonnet theorem *Adv. Math.* **213** (2007), 1–52.

[2] T.N. Bailey, M.G. Eastwood, and C.R. Graham, Invariant theory for conformal and CR geometry, *Ann. Math.* **139** (1994), 491–552.

[3] T. Branson, and A.R. Gover, Conformally invariant operators, differential forms, cohomology and a generalisation of Q-curvature, *Comm. Partial Differential Equations*, **30** (2005), 1611–1669.

[4] T. Branson, and A.R. Gover, he conformal deformation detour complex for the obstruction tensor, *Proc. Amer. Math. Soc.* **135** (2007), 2961–2965.

[5] A. Čap, Infinitesimal automorphisms and deformations of parabolic geometries, preprint math.DG/050835, *J. Europ. Math. Soc*, to appear.

[6] A. Čap, and A.R. Gover, Tractor calculi for parabolic geometries, *Trans. Amer. Math. Soc.* **354** (2002), 1511–1548

[7] A. Čap, and A.R. Gover, CR-Tractors and the Fefferman Space, arXiv:math/0611938.

[8] A. Čap, and V. Souček, Subcomplexes in Curved BGG-Sequences, arXiv:math/0508534.

[9] A. Čap, and V. Souček, Curved Casimir operators and the BGG machinery, *SIGMA Symmetry, Integrability and Geometry: Methods and Applications*, to appear, arXiv:0708.3180.

[10] A. Čap, J. Slovák, and V. Souček, Bernstein-Gelfand-Gelfand sequences, *Ann. of Math.,* (2) **154** (2001), 97–113.

[11] D.M. Calderbank, T. Diemer, Differential invariants and curved Bernstein-Gelfand-Gelfand sequences, *J. Reine Angew. Math.* **537** (2001), 67–103.

[12] Eastwood and Rice, Conformally invariant differential operators on Minkowski space and their curved analogues, *Comm. Math. Phys.,* **109**, (1987), 207–228. (Erratum: *Comm. Math. Phys.*, **144** (1992), 213.)

[13] C. Fefferman, Parabolic invariant theory in complex analysis, *Adv. Math.* **31** (1979), 131-262.

[14] C. Fefferman, and C.R. Graham, Q-Curvature and Poincare Metrics, *Math. Res. Lett.*, **9** (2002), 139–151.

[15] C. Fefferman, and C.R. Graham, The ambient metric, arXiv:0710.0919.

[16] A.R. Gover, Invariants on projective space, *Jour. Amer. Math. Soc.* **7** (1994), 145–158.

[17] A.R. Gover, C.R. Graham, CR Invariant Powers of the sub–Laplacian *J. Reine Angew. Math.* **583** (2005), 1–27.

[18] A.R. Gover, and K. Hirachi, Conformally invariant powers of the Laplacian—a complete nonexistence theorem, *J. Amer. Math. Soc.* **17** (2004), no. 2, 389–405.

[19] A.R. Gover, P. Somberg, and V. Souček, Yang-Mills detour complexes and conformal geometry, arXiv:math/0606401

[20] C.R. Graham, R. Jenne, L.J. Mason, G.A. Sparling, Conformally invariant powers of the Laplacian, I: Existence, *J. London Math. Soc.* **46**, (1992) 557–565.

[21] C.R. Graham, M. Zworski, Scattering matrix in conformal geometry, *Invent. Math.* **152** (2003), 89–118.