Ξ–Transform, Banff, March 25 – April 1 2007.

Most of the participants of the meeting had strong background in conformal geometry in one form or other. The Ξ–transform of Sparling was a unifying theme, and a lot of progress has been made in understanding the geometry of ordinary differential equations either in twistor terms, or in the language of Cartan connection.

During the meeting I made progress towards understanding the metricity of projective connection. The discussions with Gover were most useful. I have also made some connections between the theory of ASD structures in neutral signature, and the systems of second order ODEs. This is likely to result in a collaborative work with Doubrov and Grant.

The following technical comments come directly from the participants.

Maciej Dunajski

BORIS DOUBROV.

- Links between geometry of ordinary differential equations, twistor theory and conformal structures.

  According to the work of Daniel Grossman systems of 2 second order ODE's with vanishing generalized Wunschmann (or Wilczynski) invariant have a remarkable property. Their solution space has a natural anti-self dual conformal structure of split signature. This has been discussed with Maciej Dunajski with a number of explicit examples constructed. There seem to be an ultimate relation between solutions of heavenly equation and ODE's with certain higher order invariants equal to 0.

  According to works of Ezra Newman, general conformal structures of signature (1,3) and (2,2) naturally appear on solution spaces for two second order PDE's on one function of two variables with some invariants equal to 0. There seem to be a direct link between these invariants and classical projective differential geometry of surfaces in RP^3. More research is required in this direction.

- Rational curves in parabolic geometries.

  We discussed with Ben McKay how rationality of certain curves in Cartan geometries implies vanishing of certain invariants and often the flatness of the connection. The role of sl_2 subalgebras in the model algebra and various technical computational aspects were discussed. One of the open questions is how McKay's results on parabolic geometries can be generalized to more general non-parabolic cases.

- Cartan connections associated with overdetermined systems of differential equations.

  There was a number of fruitful discussions with Rod Gover on the Cartan geometries associated with (over)determined systems of finite type differential equations (either ordinary or partial). There is a number of partial results in this direction, but no complete picture up to now. There are two natural parts of this problem: construction of the

flat Cartan connection in case all compatibility conditions are satisfied, and interpreting compatibility conditions in terms of the curvature tensor of non-flat Cartan connection.

- Old and new version of Fefferman construction.

  An excellent presentation of Goerge Sparling on his vision of Fefferman constructions shows that there should be apparent generalizations of this construction for more general structures given by the identification of the tangent vector bundle with

  1. tensor product of two vector bundles (so-called almost Grassmanian structures)
  2. skew-symmetric square of a fixed vector bundle (so-called almost spinorial structures)

  According to George Sparling, the reduction of these structures by an infinitesimal symmetry has a striking similarity with the classical Fefferman construction. This has also been discussed with Maciej Dunajski in case of reductions of (2,2) conformal structures and their reconstruction from reduced data.

ROD GOVER. Differential equations and their solution spaces determine geometric structures. In some cases these structures completely characterise the equation and/or its solution space. One effective approach to extracting this data and invariants thereof is via Cartan connections or tractor connections. These are connections on prolonged differential systems, the differential system involved can come from the original equation itself or, in another direction, may be derived from the solution space of this. These techniques and comparisons of approaches was examined and also discussed with the participant Boris Doubrov. Another application looked at (and discussed with Maciej Dunajski) was the construction of invariants obstructing affine connections from being projectively related to a Levi-Civita metric connection. There is an established tractor connection related to this problem and this proved effective in recovering invariants in low dimensions. It remains to be seen if the vanishing of these invariants is sufficient to solve the problem. Further progress was made in a study of the solution space structure (and related issues) for linear operators which arise from a composition of commuting linear operators. This was subsequently publicly archived at arXiv:math/0701377.

JAMES GRANT.

- M. Dunajski and I completed the final version of our paper "Multidimensional integrable systems from deformations of Lie algebra homomorphisms" (joint with I.A.B. Strachan), which has now been submitted for publication.

- Based upon work that I presented in my talk at BIRS, I had conversations with M. Dunajski concerning the possibility of finding projective structures on hyperbolic monopoles moduli spaces. Such structures may allow an approach to the scattering theory of hyperbolic monopoles, and is a possible direction for future collaborative research.

- Based upon remarks of B. Doubrov after the talk of M. Dunajski, I became aware of the work of Grossman concerning the connection between anti-self-dual conformal structures of split signature on four-manifolds and equivalence classes of pairs of second order ordinary differential equations under point transformations. During the time at BIRS, I had several conversations with M. Dunajski and B. Doubrov concerning this correspondence, and carried out initial calculations investigating the ordinary differential equations that correspond to hyper-symplectic and Ricci-flat anti-self-dual four-manifolds. At this moment, I am investigating the symmetries of the equations governing hyper-symplectic structures, and the corresponding transformations that these symmetries generate on the space of second order differential equations modulo point transformations.

GEORGE SPARLING.

- Non-traditional applications of twistor theoretic ideas: integral geometry, projective structures, mini-twistor spaces, dispersion free equations, ODE's, perhaps leading in particular to a general theory of ODE's where one can trace the full structure through from the most elementary considerations: $F(x, y; a, b, c, ..) = 0 \rightarrow ....?$

- Applications of the same to such areas as the ODEs associated to null hypersurface twistor curves in space-time.

- Understanding of Fefferman-type constructions: conformal holonomy characterizations of Fefferman structures; generalized such structures used in connection with third order ODEs with non-vanishing Wunschmann invariant and in connection with the applications of the Xi-transform in general relativity. Setting of Fefferman-type theory in the context of parabolic geometry.

- That non-traditional signatures such as (2, 2) (self-dual Yang–Mills, gravity, Radon transform, etc), the Xi-transform (3,3) and (4, 4), F-theory (2, 10) or (3, 9) have a vital role to play.

LOUIS BOYA. The meeting was most interesting. I was prepared to discuss topics of two-times physics in relation to my own work on F-Theory. When preparing my talk, I kept in mind that F-theory is not very popular, and probably unknown to most mathematicians. In the talk I emphasis ed the current difficulties wit string theory with not many new results coming out of it, and how F theory may posssibly may provide new impact.

I interacted with Hao Xing whose research interests were similar to mine. The atmosphere was very congenial, and I have had ample discussions with most of the participants. I learned about the Xi transform, and how it lead to a model of six dimensions with split signature (3, 3) whereas in F-Theory we are dealing with 12 = (2, 10) signature. I especially enjoyed the talks by the organizers, G. Sparling and M. Dunajski, as well as some mathematicians , in particular Ben McKay.

Incidentally, I managed also to attend some talks on a parallel meeting on pure mathematics (low-dimensional topology). I do not have to talk about the surroundings, which were marvelous, with realistic possibilities of trekking and skiing. I think I was just too fortunate to be involved and being able to attend.

BEN MCKAY. During the Xi Transform workshop, I learned a lot about the geometry of ordinary differential equations, particularly from the talk of Boris Doubrov, who explained the relationship between the Wilczynski theory of linear ordinary differential equations and the work of Tresse, Wunschmann and Cartan on nonlinear ordinary differential equations. Doubrov also gave an outline of his approach to finding differential invariants for systems of higher order equations under arbitrary coordinate transformations, following a surprisingly strong analogy with the theory of second order equations and parabolic geometries. I thought some more about the problem of finding and classifying complex 2nd order ODEs whose generic solution is a closed Riemann surface. I realized during the workshop that this problem has some subtleties related to KAM theory. I also came to appreciate from talking to Maciej Dunajski and George Sparling that there are some nice relations between ordinary differential equations and split signature pseudo-Riemannian 4-manifolds, which I had not previously considered. Unfortunately, as I was just about to discover a major theorem, my head exploded from eating too much.

HAO XING. During this workshop, I discussed the possible generalization of Chern-Simons action to the F-theory context. Comparing to the Chern-Simon action in M theory, which is calculated by index of Dirac operator on 11 dimensional spacetime, we propose to define the Chern-Simon action in F-theory by a loop index on the loop space of a 12 dimensional spacetime. I discussed the loop index theory discovered by Witten in late 1980's. Based on this definition of Chern-Simons action on F-theory, I also discussed the connection among F-theory, type IIA, type IIB string theory and M theory using generalized T-duality. Moreover, I explained the difference and connection between our F-theory with the orginal F- theory proposed by Vafa, which is an elliptic fibration over 10 dimensional manifold. This talk I gave in this workshop is based on a joint work with my advisor Igor Kriz.

Moreover, I learned a lot from other people in this workshop. Luis taught me a lot on F-theory, and I told him a connection between loop index and monster group. I learned what is anti–self–dual metric from Maciej and a lot of conformal and Cartan geometry from other people.