BIRS Workshop
Resolution of singularities, factorization of birational mappings, and toroidal geometry
December 11–16, 2004

MEALS

Breakfast (Continental): 7:00 - 9:00 am, 2nd floor lounge, Corbett Hall, Sunday - Thursday
*Lunch (Buffet): 11:30 am - 1:30 pm, Donald Cameron Hall, Sunday - Thursday
*Dinner (Buffet): 5:30 - 7:30 pm, Donald Cameron Hall, Saturday - Wednesday
Coffee Breaks: As per daily schedule, 2nd floor lounge, Corbett Hall
***Please remember to scan your meal card at the host/hostess station in the dining room for each lunch and dinner.**

MEETING ROOMS

All lectures are held in the main lecture hall, Max Bell 159. *Please note that the meeting space designated for BIRS is the lower level of Max Bell, Rooms 155-159. Please respect that all other space has been contracted to other Banff Centre guests, including any Food and Beverage in those areas.*

SCHEDULE

Saturday, December 11

17:30–19:30 Dinner

Sunday, December 12

7:00–9:00 Breakfast

9:00–9:15 Welcome and introduction.

9:20–10:10 Dan Abramovich, Overview: Resolution of singularities and toroidal geometry.

10:10–10:50 Coffee

10:50–11:40 Edward Bierstone, Resolution of singularities I.

11:40–13:30 Lunch

13:30–14:20 Kalle Karu, Factorization of birational maps I.

14:30–15:20 Cinzia Casagrande, The maximal Picard number of a toric Fano variety.

15:20–16:00 Coffee

16:00–16:50 Maurice Rojas, Torsion points on algebraic sets and A-discriminants

17:30–19:30 Dinner

Monday, December 13

7:00–9:00 Breakfast

9:10–10:00 Dale Cutkosky, Toroidalization of morphisms I.

10:00–10:30 Coffee

10:30–11:20 Edward Bierstone, Resolution of singularities II.

11:30–12:20 Kalle Karu, Factorization of birational maps II.

12:20–13:30 Lunch

13:30–14:20 Shihoko Ishii, The local Nash problem for a quasi-ordinary singularity.

14:30–15:20 Bernard Teissier, What can toroidal geometry say about local uniformization?

15:20–16:00 Coffee

16:00–16:50 Mark Spivakovsky, Puiseux expansions, a local analogue of Nash's space of arcs and the local uniformization theorem.

17:30–19:30 Dinner

Tuesday, December 14

7:00–9:00 Breakfast

9:10–10:00 Edward Bierstone, Resolution of singularities III.

10:00–10:30 Coffee

10:30–11:20 Dale Cutkosky, Toroidalization of morphisms II.

11:30–12:20 Jarek Włodarczyk, Morelli's pi-desingularization and the weak factorization theorem.

12:20–13:30 Lunch

17:30–19:30 Dinner

Wednesday, December 15

7:00–9:00 Breakfast

9:10–10:00 Kalle Karu, Factorization of birational maps III.

10:00–10:40 Coffee

10:40–11:30 Dale Cutkosky, Toroidalization of morphisms III.

11:30–11:45 Group photo [1]

11:45–13:20 Lunch

13:20–14:10 Vincent Cossart, Towards local uniformization along a valuation in Artin-Schreier extensions (dimension 3).

14:20–15:10 Olivier Piltant, Applications of ramification theory to resolution of three-dimensional varieties.

15:10–15:40 Coffee

15:40–16:30 Franz-Victor Kuhlmann, What can the theory of valued fields say about local unifomization?

16:40–17:30 Anne Fruehbis-Krueger, Computer implementation of desingularization algorithms.

[1] A group photo will be taken on Wednesday at 11:30 am, directly after the last lecture of the morning. Please meet on the front steps of Corbett Hall.

17:30–19:30 Dinner

Thursday, December 16

7:00–9:00 Breakfast

11:30–13:30 Lunch

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ABSTRACTS
(in alphabetic order by speaker surname)

Abstracts of the three mini-courses (by **Edward Bierstone**, **Dale Cutkosky** and **Kalle Karu**) have been posted separately on the web page of the Workshop.

Speaker: **Dan Abramovich** (Brown University)
Title: *Resolution of singularities and toroidal geometry*
Abstract: Toric geometry is the study of the rich and beautiful geometry of toric varieties, a fairly limited class of rational varieties. Yet this subject interacts in surprising ways with the birational geometry of *arbitrary* varieties. This workshop is devoted to this point of friction between the subjects.

Hironaka's theorem on resolution of singularities [10] is beautiful and powerful, yet it's original proof was so difficult that few read it, and its original statement left a number of issues unanswered, even in charactersitic 0. Bierstone's lecture series will be devoted to demystifying *canonical resolution of singularities*, including some new state of the art ideas. An ulterior motive for a thorough understanding of resolution of singularities is given by the Toroidalization Conjecture, see below.

Canonical resolution of singularities in its strongest form is a cornerstone in the known proofs of the theorem on *weak factorization of birational maps* [AKMW, W1], the topic of Karu's lecture series. This is one point where toric geometry has so far been absolutely essential, and quite surprisingly so [W]. In addition, one major obstacle for proving Hironaka's conjecture on *strong* factorization of birational maps is the toric case - known as Oda's strong conjecture. Karu will explain the delicate issues this conjecture raises.

The conjectural *strong factorization* is a sort of resolution of singularities of a proper birational map $X \to Y$. A natural generalization is the *Toroidalization Conjecture* [AK, AKMW], whose origins go back to Akbulut and King. Here, toric geometry is less a tool and more a goal: given a projective dominant morphism $X \to Y$, construct, using unly sequences of blowings up smooth centers, modifications $X_1 \to X$ and $Y_1 \to Y$, such that the rational map $X_1 \to Y_1$ is a *toroidal morphism,* which is in a strong sense a *regular* map locally modeled on a torus equivariant map of toric varieties [KKMS]. If time permits I will dwell on this notion. I will also say a few words on the relationship between toroidalization and semistable reduction [AK], another topic where toric geometry has an important role.

The strongest results towards the Toroidalization Conjecture are due to Cotkosky [C] - [C5]. His lecture series will be devoted to his work on toroidalization of a threefold map. This is a beautiful and powerful result, yet it's original proof is extremely difficult, I venture to say more than Hironaka's. It is our hope that the combination of Birstone's and Cutkosky's lectures will help in the understanding of this result, and maybe in demystifying its proof.

References

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[C5] Toroidalization of birational morphisms of 3-folds, preprint, math.AG/0407258.

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[W] Włodarczyk, Jarosław,*Birational cobordisms and factorization of birational maps.* J. Algebraic Geom. 9 (2000), no. 3, 425–449.

[W1] Wlodarcyzk, J., *Toroidal varieties and the weak factorization theorem*, Invent. Math. 154 (2003), no. 2, 223–331.

Speaker: **Cinzia Casagrande** (Università di Pisa)
Title: *The maximal Picard number of a toric Fano variety*
Abstract: Let X be a smooth, complex Fano variety of dimension n, Picard number r and pseudo-index i. The generalized Mukai conjecture predicts that if $i > 1$, then r is at most $n/(i-1)$. We will show that when X is toric, this conjecture is true; moreover in the case $i = 1$, r is at most $2n$. The problem is equivalent to determining how many vertices a Fano polytope can have.

Speaker: **Vincent Cossart** (Université de Versailles)
Title: *Towards local uniformization along a valuation in Artin-Schreier extensions (dimension 3)*
Abstract: In this talk, I will define an invariant ω which should lead to a proof of local uniformization in an Artin-Schreier extension in dimension 3. The main topics will be the following:
(1) Definitions of adapted jacobians.
(2) Definition of ω.
(3) Permissible blowing-ups.
(4) Description of the main different cases.
Joint work with Olivier Piltant

Speaker: **Shihoko Ishii** (Tokyo Institute of Technology)
Title: *The local Nash problem for a quasi-ordinary singularity*
Abstract: In the paper on 1968,Nash introduced the Nash map and posed the Nash problem. Let X be a variety over the complex number field. Roughly speaking, the Nash map is a map from the set of

maximal irreducible families of arcs passing through the singularities of X to the set of essential divisors on resolutions of X. The Nash problem is whether this map is bijective. As Nash considered only isolated singularities cases in his preprint, the Nash problem can be understood in two ways, which coincide when the singularities is isolated. One is called just the Nash problem and the other is called the local Nash problem. The former is affirmatively answered for a toric variety by Ishii-Kollar's paper [13]. In this talk I will show you the affirmative answer for a toric variety and a quasi-ordinary singularity.

References

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Speaker: **Franz-Viktor Kuhlmann** (University of Saskatchewan)
Title: *What can the theory of valued fields say about local uniformization?*
Abstract: The problem of local uniformization can be reformulated as a problem about the structure of valued function fields. I will quickly review this reformulation.

One part of the problem is elimination of ramification. This is harder in positive characteristic because there, you also have to deal with wild ramification, and the main obstacle turns out to be the defect of valued field extensions. I will give examples of non-trivial defect. Then I will show why certain valuations (called "Abhyankar valuations") always admit a positive solution, and describe the valuation theoretical theorems used for the solution.

In order to generalize these theorems, we have to learn more about the defect. A first step is to classify Artin-Schreier extensions with non-trivial defect, which I have done in a recent preprint. Based on this classification, there is joint work in progress with O. Piltant on the higher ramification groups of such extensions. Future work shall also lead to a better understanding of defect extensions generated by additive polynomials other than the Artin-Schreier polynomial.

Speaker: **Olivier Piltant** (CNRS et Université de Versailles)
Title: *Applications of ramification theory to resolution of three-dimensional varieties.*
Abstract: In this talk, I will sketch the proof of the reduction of "resolution of singularities of threefolds over a perfect field of positive characteristic" to "local uniformization in Artin-Schreier extensions". This result was proved by Abhyankar in dimension two along his 1955 proof of resolution.

The main topics will be the following:

(1) Glueing of local uniformizations. This step relies heavily on the existence of strong factorization in dimension two.

(2) Ramification theory of general valuations. The main point here is the solvability of inertia groups (Krull's theorem).

(3) Pushing down local uniformization in cyclic extensions of prime degree distinct from the characteristic.

Joint work with V. Cossart.

Speaker: **Maurice Rojas** (Texas A&M University)
Title: *Torsion Points on Algebraic Sets and A-Discriminants*
Abstract: The A-discriminant, which arose from work of Gelfand, Kapranov, and Zelevinsky on hypergeometric functions, is central in algorithmic toric geometry. For instance, the A-discriminant contains all known multivariate resultants, including the toric resultant, as special cases. More geometrically, it is the defining polynomial for the variety projectively dual to the toric variety corresponding to a point set A. We detail the following new results:

(1) Suppose A is a circuit, i.e., a set of $n+2$ affinely dependent points in Z^n with every proper subset affinely independent. Then we can compute the sign of the A-discriminant in polynomial time.

(2) A subexponential algorithm for deciding whether an algebraic set contains a torsion point in the algebraic torus $(C^*)^n$. (For the special case of a hypersurface, this is the same as deciding the vanishing of a certain A-discriminant, where the convex hull of A is the convex hull in R^{2n-1} of an n-zonotope and an n-polytope.)

(3) Under a plausible number theoretic hypothesis strictly weaker than GRH, we have a subexponential algorithm for deciding the vanishing of A-discriminants in complete generality.

In all the above cases, no algorithm of subexponential complexity was previously known. Our techniques are completely different from the usual Grobner basis, resultant, or exterior algebraic methods. We also state some connections to real algebraic geometry.

Speaker: **Mark Spivakovsky** (Université Paul Sabatier, Toulouse)
Title: *Puiseux expansions, a local analogue of Nash's space of arcs and the local uniformization theorem.*
Abstract: The main purpose of this talk is to introduce a new mathematical object, which we call the local analogue of Nash's space of arcs. This local space of arcs is associated to a valuation centered in a local noetherian ring. The usual formal power series which parametrize an arc in Nash's sense are replaced by power series with exponents in the (non-well ordered) set of non-negative elements in the first isolated subgroup of the value group of the given valuations. The most immediate applications are to the local uniformization theorem, but in the long run the local space of arcs should help understand the precise connection between resolution of singularities and the classical Nash's arc space, which is every singularist's dream.

Speaker: **Bernard Teissier** (CNRS es Université de Paris VII)
Title: *What can toroidal geometry say about local uniformization?*
Abstract: Over an algebraically closed field, the problem of local uniformization can be reformulated as a problem about the structure of the specialization of an excellent equicharacteristic local ring R, with algebraically closed residue field, to the graded algebra associated to a valuation which is "rational" (no residue field extension from R to the valuation ring). This graded algebra corresponds to a toric variety, possibly of infinite embedding dimension, and a suitable partial resolution of its singularities should extend to a uniformization of the extension of the valuation to a suitable completion of R, from which one gets by excellence the uniformization of the valuation on R.

This approach avoids the difficulties related to ramification, but meets new ones related to completion and infinite dimensional phenomena, which will be discussed.

Speaker: **Jarek Włodarczyk** (Purdue University)
Title: *Morelli's pi-desingularization and the weak factorization theorem*
Abstract: We give the sketch of the proof of the Weak factorization theorem with particular emphasis on the pi-desingularization of birational cobordisms.