Rank Conjectures in Algebraic Topology and Commutative Algebra

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11 September 2022 – 16 September 2022

1 Overview of the Field

The goal of the workshop was to bring together researchers from mainly two areas: algebraic topology and commutative algebra. Among the numerous points of contact between them, the focus was on certain long open, fundamental, and closely related conjectures that postulate strong lower bounds on total homology of objects appearing naturally in each setting. An outstanding one on the topological side is Halperin's Toral Rank Conjecture that if a d-dimensional torus acts freely on a compact topological space, then the total rank of its rational cohomology must be at least 2^d. According to Halperin himself this problem has been around since the late 1960s. Carlsson (1980s) conjectured a similar lower bound if the space admits a free action of $(\mathbb{Z}/p)^d$, an elementary abelian p-group of rank d, where p is a prime number, and homology is taken with coefficients in $\mathbb{F}_p$. In the same context, Benson and Carlson (1987) conjectured that a finite group acts freely on some product of r spheres if and only if its rank is at most r. This builds on the work of Browder from the 1980s and is connected to developments in the cohomology of finite groups, due to Benson and Carlson.

On the commutative algebra side, the objects of interest are the free resolutions of modules of finite length (the condition is analogous to the space being compact) over a polynomial ring, $R = k[x_1, \ldots, x_d]$, and more generally, over a regular local ring of dimension d. The Hilbert Syzygy Theorem (1880s) states that such modules have a minimal free resolution of length at most d, whereas work of Auslander and Buchsbaum, and Serre (1960s) says that it cannot be less than d. The ranks of the free modules in the minimal resolution, called Betti numbers, are fundamental invariants, for they reflect many geometric properties, such as multiplicities of singularities and intersection numbers. Buchsbaum and Eisenbud, and independently, Horrocks, conjectured (around 1977) that the ith Betti number of a module should be at least $\binom{d}{i}$. Avramov (around 1985) made the connection to Halperin's conjecture and asked if at least the total Betti number has to be at least 2^d; this is called the Total Rank Conjecture.

There is more than just a numerical coincidence between the conjectures. Indeed, as is their wont, by training, algebraic topologists have formulated algebraic analogues of the conjectures of Halperin and Carlsson that imply the conjectures in commutative algebra, and also their topological counterparts. There is one point worth noting: These conjectures concern complexes of modules, and not just modules; this is important for applications to topology. Thanks to the contribution of many researchers, these conjectures have been settled for small values of d; mainly $d \leq 3$.

Notably, for $p = 2$ Carlsson used a version of Bernstein-Gelfand-Gelfand correspondence (which he rediscovered in this special case) to reformulate his p-toral rank conjecture as a problem about differential modules over the polynomial ring R, that would also imply the Total Rank Conjecture. Carlsson solved this algebraic conjecture for $d \leq 3$. For $p > 2$, notable contributions come also from the work of Adem, Allday, Baumgartner, Browder, Puppe, Swan, and also Carlsson himself (on the topological side), and Avramov, Buchweitz, and Iyengar (on the commutative algebra side). The algebraic analogue of Halperin's conjecture has to do with differential graded modules over the polynomial ring R, now over the field $\mathbb{Q}$, viewed as a differential graded algebra. The translation is via the Borel construction, which can be seen as a topological avatar of the Bernstein-Gelfand-Gelfand correspondence. Carlsson's conjecture was verified by Adem and Browder for actions on products of equidimensional spheres for p odd. Adem and Smith established the conjecture of Benson and Carlson for finite p-groups of rank 2. Their methods show, more generally, that if a finite group acts on a sphere with rank one isotropy, then it acts freely on a finite complex homotopy equivalent to a product of two spheres. However we still do not have a complete characterisation of groups that can act freely on such a product. The methods of homotopy group actions should play a role here. Besides these, on the whole sporadic, results, the conjectures remained open in general.

2 Recent Developments

Two remarkable developments have taken place in the last few years: Walker (2016) proved the Total Rank Conjecture for modules over polynomials rings; with one caveat, the field should **not** have characteristic 2. In fact his results are more general and cover complexes. Curiously the key tool in Walker's proof is Adams operations in K-theory, which were developed by the topologists, and is a part of their standard tool-kit. Subsequently, Amann and Zoller used Walker's proof of the Total Rank conjecture to establish new cases of Halperin's conjecture.

Another noteworthy development was that Iyengar and Walker (2018) found counterexamples to many of the algebraic versions of the conjectures for complexes made by Adem, Browder, Swan, and Carlsson. These examples are constructed using "Lefschetz elements" in exterior algebras. The existence of such elements has been known over fields of characteristic 0 for almost a 100 years, and is a key input in the proof of the "Hard Lefschetz theorem". The positive characteristic analogue was proved only a few years ago by Conca, Iyengar, and Herbig (2018), and this case is crucial for the counter-examples connected with Carlsson's conjecture. Once again all this comes with the same caveat as before: the examples do not cover characteristic 2!

The examples constructed by Iyengar and Walker have led to a revival of interest in the conjectures. On the one hand, it seems to be that the conjectures arising from topology are truly statements in topology, and not amenable to a pure algebraic treatment, except, perhaps, in characteristic 2. Indeed, Rüping and Stephan (2019) have shown that these counterexamples cannot arise from topological constructions. All this has rekindled some of the dialogue between the two areas started long ago by Avramov and Halperin. This has also brought to the surface the question of what algebraic properties correspond to a complex stemming from topology.

3 Final program: schedule, titles and abstracts

Schedule

	Monday	Tuesday	Wednesday	Thursday	Friday
9:00-10:00	Adem I	Adem II	Walker I	Şentürk	Rüping
10:00-10:30	Break	Break	Break	Break	Eisenbud
10:30-11:30	Hanke I	Hanke II	Walker II	Open mic	
11:30-1:00					
1:00-2:00	Tour			Franz	
2:00-3:00	Peeva I (20 min shift)	Peeva II		Erman	
3:00-3:30	Break	Break		Break	
3:30-4:30	Elevator pitches	Elevator pitches		Working groups	
4:30-5:30	Working groups	Working groups			

Lectures

Alejandro Adem: Finite group actions, cohomology of groups, and rank conjectures

Lecture 1 : We will review basic facts about transformation groups and the resulting restrictions on groups that can act freely on a finite CW-complex. Cohomological methods will be introduced and geometric examples will be provided. We will consider the relationship between the rank of the group and the cohomology of a space on which it acts freely. We will review the work of Carlsson, Browder and others on this problem, motivated by the case of a product of spheres.

Lecture 2: In this talk we will consider the problem of constructing group actions with prescribed isotropy on a finite complex with a fixed homotopy type. This will involve methods from representation theory and homotopy theory. For rank one groups this builds on the classical characterization of groups acting freely on spheres, leading to a generalized notion of cohomological periodicity with a corresponding geometric characterization. We will discuss extensions to groups of higher rank.

Bernhard Hanke: Rational and tame homotopy theory

Lecture 1: Rational homotopy theory. Sullivan-de Rham theorem, small cochain models via Postnikov decomposition, examples, cochain models for torus actions.

Lecture 2: Tame homotopy theory. Cenkl-Porter theorem, tame Hirsch lemma, cochain models for p-torus actions, the stable free rank of symmetry of products of spheres.

Irena Peeva: Survey on Hilbert functions and Betti numbers in the Artinian case

We will cover the following topics:

1. Motivation and Basic Definitions of Hilbert functions, Free resolutions, and Betti numbers.
2. Formulas.
3. Upper Bounds.
4. Lower Bounds.
5. The Buchsbaum–Eisenbud–Horrocks Conjecture.
6. A related conjecture about Ext.
7. Other problems about resolutions in the artinian case.
8. Conjectures about Hilbert functions in the artinian case.

Mark Walker: Rank conjectures in algebra

I will first discuss the "Total Rank Conjecture" in commutative algebra, which is a weak form of the famed Buchsbaum-Eisenbud-Horrocks Conjecture regarding lower bounds on the Betti numbers of modules of finite projective dimension. The discussion will include both the proof of the Total Rank Conjecture in some cases (due to myself) and counter-examples to a generalized version of it (due to S. Iyengar and myself). I will explain the connection of these results with the Toral Rank Conjecture in topology, and, in particular, I will discuss why the positive results in algebra fail to prove the Toral Rank Conjecture, and why the counter-examples do not disprove it. The work of Iyengar and myself is also related to a conjecture of Carlsson regarding free actions of elementary abelian p-groups on topological spaces; specifically, we produce counter-examples to a purely algebraic generalization of this conjecture. I will discuss these examples and why, as before, they don't disprove Carlsson's original conjecture.

Talks

David Eisenbud: Summands in High Syzygies

Work on infinite resolutions beyond the cases of complete intersections and Golod rings has tended to focus on the sequence of Betti numbers. Hai Long Dao and I have recently begun to study a question of a different kind, and I will report on this joint work: Let R = S/I be an artinian quotient of a regular local ring S, with residue field k. When does it happen that k is a direct summand of a syzygy module in the R-free resolution of k, or indeed in the R-free resolution of every module? We were surprised by what we found experimentally, and were able to prove a little of what we observed.

Daniel Erman: Generic matrix factorizations

I'll discuss the Buchweitz-Greuel-Schreyer Conjecture on the minimal rank of a matrix factorization, and my recent proof that the conjecture holds for generic polynomials.

Matthias Franz: Syzygies in equivariant cohomology

Syzygies are a notion from commutative algebra that interpolates between torsion-free, reflexive and free modules. We discuss an application of syzygies to equivariant cohomology.

Let X be a "nice" space with an action of a torus T of rank n, and take cohomology with rational coefficients. By a result of Chang-Skjelbred, the sequence

$$0 \to H_T^*(X) \to H_T^*(X^T) \to H_T^{*+1}(X_1, X^T)$$

is exact if $H_T^*(X)$ is a free module over $R = H^*(BT)$. Here X_1 denotes the orbits of dimension at most 1. This significantly simplifies the computation of $H_T^*(X)$ and forms the basis of the so-called GKM method. The CS sequence above can be extended to the Atiyah-Bredon sequence

$$0 \to H_T^*(X) \to H_T^*(X_0) \to H_T^{*+1}(X_1, X_0) \to H_T^{*+2}(X_2, X_1) \to \\ \cdots \to H_T^{*+n-1}(X_{n-1}, X_{n-2}) \to H_T^{*+n}(X_n, X_{n-1}) \to 0,$$

where $X_k \subset X$ denotes the subset of orbits of dimension at most k.

It turns out that the AB sequence is exact at the first k terms if and only if $H_T^*(X)$ is a k-th syzygy over R. In particular, the CS sequence is exact if and only if $H_T^*(X)$ is a reflexive R-module. If X satisfies Poincaré duality, then this is also equivalent to the perfection of the equivariant Poincaré pairing.

(This is joint work with Chris Allday and Volker Puppe.)

Henrik Rüping: Steenrod closed parameter ideals in $H^*(BA_4; \mathbb{F}_2)$

In this talk I will report on recent results with Ergün Yalcin and Marc Stephan providing obstructions to the existence of free A_4 actions on a product of two spheres. For such an action we can look at the map in cohomology induced by the classifying map. Its kernel is a Steenrod closed parameter ideal in $H^*(BA_4)$. We provide a full classification of these ideals. It turns out that these ideals are sparse. While the question of which ideals can be realized this way is interesting, much less is known there.

Berrin Şentürk: An algebraic approach to Rank Conjecture with examples of small rank

A long-standing Rank Conjecture states that if an elementary abelian p-group acts freely on a product of spheres, then the rank of the group is at most the number of spheres in the product. In this talk, we will discuss the algebraic version of the Rank Conjecture given by Carlsson for a differential graded module M over a polynomial ring. Then we consider the varieties of square-zero upper triangular matrices corresponding to the differentials of such modules. By stratifying these varieties via Borel orbits and imposing conditions coming from the algebraic conjecture, we state a stronger conjecture about varieties of matrices. Using the corresponding free flag construction, we show that $(\mathbb{Z}/2\mathbb{Z})^4$ cannot act freely on a product of 3 spheres of any dimensions.

4 Fostering interactions

Reading Seminar Online: rank conjectures

On the previous months to the workshop, a weekly online seminar was organized where young participants gave lectures on the topics of the conference. The seminar covered the central problems concerning rank conjectures in equivariant topology (Halperin's toral rank conjecture, Carlsson's p-toral rank conjecture, and the rank conjecture of Benson and Carlson) and commutative algebra. The goal of this activity was to setup the basic knowledge for both the topology and algebra side of the conjectures. Almost all participants gave a lecture about suggested topics. Here is the final schedule:

1. 29th June: Natàlia Castellana, Introduction to classical conjectures following [**?**].
2. 6th July: Clover May, Work of Adem [**?**].
3. 13th July: Maya Banks. Work of Avramov, Iyengar [**?**].
4. 20th July: Claudia Miller, Work of Walker [**?**].
5. 27th July: Marc Stephan, Work of Carlsson [**?**], [**?**],[**?**].
6. 3rd August: Foling Zou, Work of Allday, Halperin, Puppe following [**?**, Chapter 7] of Félix, Oprea, Tanré.
7. 10th August: James Cameron, Work of Adem-Smith [**?**].
8. 17th August: Ben Briggs, Work of Iyengar, Walker [**?**] .
9. 24th August: Josh Pollitz, Work of Avramov, Buchweitz, Iyengar, Miller [**?**].

Open mic session

The open mic session took place on Thursday once the lecture series finished. Online participants joined and it led to a discussion where new potential directions for strategies showed up like techniques from chromatic equivariant homotopy theory. It was an overall successful session. The discussion there inspired, among other things, the recent work of Jon Carlson, reported later in this document.

Working groups

Two type of working groups were scheduled: in-person and online. The in-person working groups where organized in a way that participants were mixed and split in two different groups whose composition vary from one day to another, in order to promote more interaction among participants. Most of the discussion on the in-person working groups was focused on the techniques and the role of G-equivariant CW-complexes to motivate the algebraic versions of the rank conjectures from the topological ones. Another interesting talking point, was an explanation of how the Buchweitz-Greuel-Schreyer Conjecture implies the Banff $\sqrt{2}$ Rank Conjecture mentioned in the outcomes below. Our impression is that the in-person workings groups were quite productive.

The online working groups were scheduled in two different time zones. In contrast with the in-person events, these online groups did not work as expected, with few participating.

Elevator pitches

There were approximately 14 fast talks (5 minutes each) given by volunteers in the Elevator Pitch sessions. These enabled participants, especially the younger ones whom we encouraged especially to speak and who made up the majority of these speakers, to present their interests and work to participants in the workshop. In some cases, this generated discussions outside of the session and it certainly made everyone aware of others for potential future collaborations.

5 Outcome of the Meeting

Our main goal for the meeting was to spark conversations between researchers in algebraic topology and commutative algebra. We were pleasantly surprised to find that in fact these conversations and the interactions, both online and in person, have already led to at least three articles:

- The manuscript *The ranks of homology of complexes of projective modules over finite groups*, by Jon Carlson, grew out of the open mic session, and follow up conversations. It builds on the work of Iyengar and Walker, to produce a whole slew of new counterexamples to the algebraic version of some of Carlsson's rank conjectures.

- The manuscript *Elliptic spaces with maximal toral Rank* by Greg Lupton and Mark Walker where the authors study the Toral Rank Conjecture and the Hilali Conjecture for elliptic spaces, with a particular focus on those elliptic spaces whose toral rank is as large as possible.
- The manuscript *The total rank conjecture in characteristic two*, by Keller VandeBogert and Mark Walker, where they prove the total Betti number conjecture of Avramov over all rings of characteristic two. This is the only case not covered by earlier work of Walker's.

In addition, the organizers in collaboration with Leopold Zoeller (who is a post-doc at the Ludwig Maximilians University in Munich, and an expert on the geometric aspects of group actions) and Mark Walker are planning to write a survey article on the rank conjectures in algebra and topology, with a focus on recent developments on this topic. The document will be aimed to be accessible to both algebraic topologists and commutative algebraists. It was clear from our experience at the meeting that there is an audience to such a document. This was prompted by a suggestion by Alejandro Adem at the Banff meeting.

Problems

We mention three notable problems from talks and discussions.

First, the p-toral rank conjecture for products of spheres says that $(\mathbb{Z}/p)^d$ can not act freely on a product of less than d spheres. Hanke made the stronger conjecture for p odd that at least d odd-dimensional spheres are required for a free action.

Second, while the bound 2^d in the Total Rank Conjecture of Avramov does not hold for complexes, Walker conjectured a weaker bound: for finite free complexes F over the polynomial ring $R = k[x_1, \dots, x_d]$ with $0 < \dim_k H(F) < \infty$, the rank of F is at least $(\sqrt{2})^d$. Erman christened this statement *the Banff* $\sqrt{2}$ *Rank Conjecture*!

Third, here is a problem by Adem: Under what conditions on the cohomology of a connected CW-complex X, does there exist a fibration $F \to E \to X$ where F has the cohomology of a product of d spheres and E is homotopy equivalent to a finite dimensional complex?

Follow-up meeting?

The enthusiasm generated by the meeting in Banff has encouraged us to consider a follow-up meeting, in a few years time. The tentative plan is to organize a two-week long workshop, possibly at the Centre de Recerca Matemàtica, in Barcelona.

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