Modular forms and Quantum knot invariants

Kazuhiro Hikami (Kyushu University),
Jeremy Lovejoy (CNRS, Université Paris 7),
Robert Osburn (University College Dublin)

March 11–16, 2018

1 Overview

A modular form is an analytic object with intrinsic symmetric properties. They have enjoyed long and fruitful interactions with many areas in mathematics such as number theory, algebraic geometry, combinatorics and physics. Their Fourier coefficients contain a wealth of information, for example, about the number of points over the finite field of prime order of elliptic curves, K3 surfaces and Calabi-Yau threefolds. Most importantly, they were the key players in Wiles' spectacular proof in 1994 of Fermat's Last Theorem. Quantum knot invariants have their origin in the seminal work of two Fields medalists, Vaughn Jones in 1984 on von Neumann algebras and Edward Witten in 1988 on topological quantum field theory. The equivalence between vacuum expectation values of Wilson loops (in Chern-Simons gauge theory) and knot polynomial invariants (for example, the Jones polynomial, HOMFLY polynomial and Kauffman polynomial) is a major source of inspiration for the development of new knot invariants via quantum groups.

Over the past two decades, there have been many tantalizing interactions between quantum knot invariants and modular forms. For example, quantum invariants of 3-manifolds are typically functions that are defined only at roots of unity and one can ask whether they extend to holomorphic functions on the complex disk with interesting arithmetic properties. A first result in this direction was found in 1999 by Lawrence and Zagier [17] who showed that Ramanujan's 5th order mock theta functions coincide asymptotically with Witten-Reshetikhin-Turaev (WRT) invariants of Poincaré homology spheres. More recently, Zagier has found experimental evidence of modularity properties of a new type for the Kashaev invariants of knots. This "modularity conjecture" implies one of the major outstanding open problems in this field, namely the Volume conjecture. This latter conjecture relates the value at $e^{2\pi i/N}$ of the Nth Kashaev invariant of a knot, or equivalently the

Nth colored Jones polynomial, to its hyperbolic volume and, if true, would give striking relations between hyperbolic geometry, quantum topology and modular forms. Other examples of such intriguing interactions can be found in the construction of new mock theta functions from WRT invariants [3], [14], the stability of the coefficients of the colored Jones polynomial [1], [6], [9] and generalized quantum modular q-hypergeometric series [13].

Due to recent rapid developments on new theories of modular forms including mock modular forms, vector-valued modular forms and quantum modular forms and their appearance in quantum topology and physics, there was a strong desire amongst researchers with diverse backgrounds to hold a workshop in order to build further connections between previously believed disparate areas.

In addition to enhancing our knowledge of recent scientific progress, our focus was to create opportunities for junior participants and to ensure appropriate representation of women. For example, 8 of the 35 participants (and 5 of the 19 speakers) were women.

2 Scientific content of the meeting

This workshop showcased recent developments and open problems concerning modular forms and quantum knot invariants. All speakers were encouraged to have their talk videotaped and notes or slides posted online. In the following, we briefly summarize some highlights of each talk. Junior speakers are indicated with a bullet ($\bullet$). Speakers from under-represented groups are designated with a star ($\star$).

Abhijit Champanerkar ($\star$) (College of Staten Island and the CUNY Graduate Center) spoke on *Mahler measure and the Vol-Det conjecture*. For a hyperbolic link in the 3-sphere, the hyperbolic volume of its complement is an interesting and well-studied geometric link invariant. Similarly, the determinant of a link is one of the oldest diagrammatic link invariants. Previously, the speaker studied the asymptotic behavior of volume and determinant densities for alternating links, which led to conjecturing a surprisingly simple relationship between the volume and determinant of an alternating link, called the Vol-Det Conjecture. This talk outlined an interesting method to prove the Vol-Det Conjecture for infinite families of alternating links using a variety of techniques from the theory of dimer models, Mahler measures of 2-variable polynomials and the hyperbolic geometry of link complements in the thickened torus.

Thomas Creutzig (University of Alberta) spoke on *Modularity and tensor categories for affine vertex algebras at admissible level*. A well-known result is that modules of a rational vertex algebra form a modular tensor category and that the modular group action on graded traces coincides with the categorical one. Prime examples are affine vertex algebras at positive integer level. This talk explained the state of the art for affine vertex algebras at admissible level and mainly restricted to the case of $sl(2)$. From the character point of view three types of traces arise: vector-valued modular forms, meromorphic Jacobi forms and formal distributions. There are also three types of categories one can associate to the affine vertex algebra and categorical action of the modular group seems to coincide with the one on characters.

Renaud Detcherry ($\bullet$) (Michigan State University) spoke on *Quantum representations and monodromies of fibered links*. According to a conjecture of Andersen, Masbaum and Ueno, the Witten-Reshetikhin-Turaev quantum representations of mapping class groups send pseudo-Anosov mapping classes to infinite order elements, when the level is big enough. This talk discussed how to relate this conjecture to a properties about the growth rate of Turaev-Viro invariants, and derive infinite families of pseudo-Anosov mapping classes that satisfy the conjecture, in all surfaces with n boundary components and genus $g > n \geq 2$. These families are obtained as monodromies of fibered links containing some specific sublinks. For further details, see [7].

Amanda Folsom ($\star$) (Amherst College) spoke on *Quantum Jacobi forms*. This talk introduced the notion of a quantum Jacobi form, marrying Zagier's notion of a quantum modular form with that of a Jacobi form. The speaker also offered a number of two-variable combinatorial generating functions as first examples of quantum Jacobi forms, including certain rank generating functions studied by Bryson-Ono-Pitman-Rhoades, Hikami-Lovejoy, and Kim-Lim-Lovejoy. These combinatorial functions are also duals to partial theta functions studied by Ramanujan. Additionally, it was shown that all of these examples satisfy the stronger property that they exhibit mock Jacobi transformations in $\mathbb{C} \times \mathbb{H}$ as well as quantum Jacobi transformations in $\mathbb{Q} \times \mathbb{Q}$. Finally, the talk discussed applications of these quantum Jacobi properties which yield new, simple expressions for the aforementioned combinatorial generating functions as two-variable polynomials when evaluated at pairs of rational numbers, and yield similarly simple evaluations of certain Eichler integrals. This is joint work with Bringmann. For further details, see [4].

Stavros Garoufalidis (Georgia Tech) spoke on *A meromorphic extension of the* $3D$*-index*. The 3D-index of Dimofte-Gaiotto-Gukov is a collection of q-series with integer coefficients which is defined for 1-efficient ideal triangulations, and gives topological invariants of hyperbolic manifolds, in particular counts the number of genus 2 incompressible and Heegaard surfaces. The talk discussed an extension of the 3D-index to a meromorphic function defined for all ideal triangulations, and invariant under all Pachner moves. This is joint work with Rinat Kashaev. For further details, see [10].

Frank Garvan (University of Florida) spoke on *Higher order Mock theta conjectures*. The Mock Theta Conjectures were identities stated by Ramanujan for his so called fifth order mock theta functions. Andrews and Garvan showed how two of these fifth order functions are related to rank differences mod 5. Hickerson was first to prove these identities and was also able to relate the three Ramanujan seventh order mock theta functions to rank differences mod 7. Based on work of Zwegers, Zagier observed that the two fifth order functions and the three seventh order functions are holomorphic parts of real analytic vector modular forms on $\mathrm{SL}_2(\mathbb{Z})$. Zagier gave an indication how these functions could be generalized. This talk detailed these generalizations and showed how Zagier's 11th order functions are related to rank differences mod 11.

Sergei Gukov (California Institute of Technology) spoke on $\hat{Z}_a(q)$. This talk discussed various ways to define and compute new q-series invariants that have integer powers and integer coefficients. After a quick review of the physical framework, the talk showed how it explains and generalizes the observations of Lawrence-Zagier and Hikami et. al. to arbitrary 3-manifolds. The talk also discussed a modular tensor category which is responsible for the modularity properties of $\hat{Z}_a(q)$. There are many unexpected and intriguing connections with various counting problems as well as with the works of Beliakova-Blanchet-Le and Garoufalidis-Le. For further details, see [11].

Shashank Kanade (•, $\star$) (University of Denver) spoke on *Some new q-series conjectures*. Representation theory of affine Lie algebras and more generally, vertex operator algebras, leads to interesting q-series identities. The q-series related to VOAs often exhibit deep modularity properties. Some of these q-series may also be of interest to the knot theorists. This talk presented some new q-series conjectures and is joint work with Matthew Russell and Debajyoti Nandi. For further details, see [15].

Rinat Kashaev (Université de Genève) spoke on *Quantum dilogarithms over gaussian groups and punctured surface mapping class group representations*. A gaussian group is a Pontryagin self-dual locally compact abelian group together with a fixed gaussian exponential that is a symmetric second order character associated with a non-degenerate self-pairing. This talk explained how a quantum dilogarithm can be used for construction of projective unitary representations of the mapping class groups of punctured surfaces of negative Euler characteristic.

Ruth Lawrence ($\star$) (Hebrew University) spoke on *Higher depth quantum modular forms from sl_3 quantum invariants*. This talk presented work-in-progress on higher depth quantum modular forms arising from sl_3 WRT invariants of the Poincare homology sphere, following the work of Bringmann.

Christine Lee (•, $\star$) (University of Texas at Austin) spoke on *Understanding the colored Jones polynomial via surfaces in 3-manifolds*. Quantum link invariants lie at the intersection of hyperbolic geometry, 3- dimensional manifolds, quantum physics, and representation theory, where a central goal is to understand its connection to other invariants of links and 3-manifolds. This lecture introduced the colored Jones polynomial, an important quantum link invariant and discussed how studying properly embedded surfaces in a 3-manifold provides insight into the topological and geometric content of the colored Jones polynomial in view of the Slope Conjectures by Garoufalidis and Kalfagianni-Tran and the Coarse Volume Conjecture by Futer-Kalfagianni-Purcell, and we will explore the potential connection to the number-theoretic properties of the polynomial. In particular, the speaker indicated how their recent joint work with Garoufalidis and van der Veen on the Slope Conjectures for Montesinos knots is related to the existence of multiple tails of the polynomial for Montesinos knots.

Antun Milas (SUNY Albany) spoke on *False/Mock/Quantum modular forms and Vertex algebras*. An important problem in conformal field theory is to describe modular transformation properties of characters of representations of a vertex algebra. By now this is well-understand for "nice" vertex algebras (e.g., if the category of modules is semisimple). But most vertex algebras have non-semisimple category of representations so modular properties of their characters is difficult to formulate. This talk focused on a family of W-algebras coming from certain extensions of affine W-algebras. Their irreducible characters have recently been proposed and studied. The speaker explained two approaches to modular invariance and Verlinde formula for their modules, both based on iterated integrals of modular forms. In the first approach, modular invariance is formulated with the help of regularized variables. In the second approach, one replaces the characters with better behaved non-holomorphic integrals. This approach is intimately linked to mock and quantum modular forms. Finally, it was discussed how the two approaches can be understood from a unified viewpoint. For further details, see [5].

Jun Murakami (Waseda University) spoke on *Presentation of knots by a braided Hopf algebra*. The fundamental group of a knot complement is called a knot group. A way to present a knot groups is the Wirtinger presentation, which is given by a conjugation action at each crossing of the knot. This presentation is also given by a conjugate quandle, which matches well to the Hopf algebra structure of the group ring of the knot group. This talk introduced the braided conjugate quandle corresponding to the braided Hopf algebra, which is a deformation of a Hopf algebra. A typical example of the braided Hopf algebra is the braided $SL(2)$ introduced by S. Majid, and so it may give a q-deformation of a $SL(2)$ representation of the knot group. This is joint work with Roland van der Veen.

Satoshi Nawata (•) (Fudan University) spoke on *Large N duality of refined Chern-Simons invariants*. Refined Chern-Simons invariants of torus knots can be defined by using modular matrices associated to Macdonald polynomials or DAHA, generalizing colored quantum invariants. The theory was originally defined in string theory, and conifold transition in string theory leads to a positivity conjecture of refined Chern-Simons invariants of torus knots. This conjecture connects refined Chern-Simons theory to enumerative geometry.

Toshie Takata ($\star$) (Kyushu University) spoke on *The strong slope conjecture for twisted generalized Whitehead doubles*. The slope conjecture proposed by Garoufalidis asserts that the degree of the colored Jones polynomial determines a boundary slope, and its refinement, the Strong Slope Conjecture proposed by Kalfagianni and Tran asserts that the linear term in the degree determines the topology of an essential surface that satisfies the Slope Conjecture. This talk presented a proof of the strong Slope Conjecture for a twisted, generalized Whitehead double of a knot K whenever K satisfies the strong Slope Conjecture and certain extra conditions. This is joint work with Kenneth L. Baker and Kimihiko Motegi.

Roland van der Veen (•) (Universiteit Leiden) spoke on *q-series coming from extending Turaev-Viro into the unit disk*. The Turaev-Viro invariant is a 3-manifold invariant (TQFT) defined at roots of unity q using a handle decomposition. Following Frohman and Kania Bartoszynska, this talk considered extending the variable q into the unit disk and discussed under what conditions one still gets an invariant and give some explicit examples of the ensuing q-series.

Katherine Walsh ($\bullet$, $\star$) (University of Connecticut) spoke on *Patterns and higher order stability in the colored Jones polynomial*. This talk discussed the patterns and stability in the coefficients of colored Jones polynomial. While much work has been done looking at the leading sequence of coefficients, the speaker emphasized work moving towards understanding the middle coefficients. This included small steps like looking at the second N coefficients of the Nth colored Jones polynomial and larger steps like looking at the growth rate of the coefficients and looking at the patterns present in the coefficients under various re-normalizations. For further details, see [20].

Paul Wedrich ($\bullet$) (Australian National University) spoke on *Knots and quivers, HOMFLY and DT*. Physicists have long been arguing that gauge theories at large rank are related to topological string theories. As a concrete example, this talk described a correspondence between the colored HOMFLY-PT polynomials of knots and the motivic DT invariants of certain symmetric quivers, which was recently proposed by Kucharski-Reineke-Stosic-Sulkowski. The speaker outlined a proof of this correspondence for 2-bridge knots and then speculated about how much of the HOMFLY-PT skein theory might carry over to the realm of DT quiver invariants. This is joint work with Marko Stosic. For further details, see [19].

Tian Yang ($\bullet$) (Texas A&M) spoke on *Volume conjectures for Reshetikhin-Turaev and Turaev-Viro invariants*. Supported by numerical evidence, Chen and the speaker conjectured that at the root of unity $e^{2\pi\sqrt{-1}/r}$, instead of the usually considered root $e^{\pi\sqrt{-1}/r}$, the Turaev-Viro and the Reshetikhin-Turaev invariants of a hyperbolic 3-manifold grow exponentially with growth rates respectively the hyperbolic and the complex volume of the manifold. This revealed a different asymptotic behavior of the relevant quantum invariants than that of Witten's invariants (that grow polynomially by the Asymptotic Expansion Conjecture), which may indicate a different geometric interpretation of those invariants than the $SU(2)$ Chern-Simons gauge theory. This talk introduced the conjecture and showed further supporting evidence, including recent joint work with Detcherry-Kalfagianni. For further details, see [8].

3 Outcome of the Meeting

The workshop was well-attended by researchers at different career levels and from a variety of countries: Canada, USA, Australia, China, Japan, Ireland, Switzerland, Israel, Germany, the Netherlands and France. After the workshop, we received extremely positive feedback, like

> *"Thank you for a wonderful workshop!"*

or

> *"It was an excellent line-up of speakers."*

Regarding the interdiscplinary nature of the workshop, one participant wrote,

"This was a wonderful workshop which brought together number theorists and knot theorists studying various forms of q-series. It was very helpful to get the other perspective on research and talk to many experts in both fields. These conversations gave me a good idea of what people in both areas are interested in and which directions I should extend my research."

The workshop also led to a number of new results, some of which were even announced before the end of the week! For example, it was generally expected that for every knot there exists a p such that for sufficiently large N, the first N coefficients of the N-colored Jones polynomial only depend on the class of N mod p. These stabilized coefficients are known as the "tail" of the colored Jones polynomial and much work has gone into studying such tails. During the workshop, Lee and van der Veen announced the surprising result that there is an infinite family of knots that does not have this property; instead the first few coefficients grow linearly in N. They proposed calling these knots Manx knots after a breed of tailless cats. The simplest known Manx knot is the pretzel knot $P(-3, 5, 5)$.

Other new results established during the workshop were reported by Detcherry, Kalfagianni, and Yang. They proved that at appropriate roots of unity, the Turaev-Viro invariants of an infinite family of links, including the "fundamental shadow links," grow exponentially with growth rate the hyperbolic volume of the link complement. This verifies the Volume Conjecture by Chen and Yang of the Turaev-Viro invariants for these links.

Finally, the workshop perfectly complemented recent activities such as the Workshop on "Low-dimensional topology and number theory" at the Mathematisches Forschungsinstitut Oberwolfach, August 20–26, 2017. We are very grateful for the opportunity to organize such a workshop and hope to continue its success with another 5-day workshop at Banff in 2020 or 2021.

4 Future Directions

The goal of this intense five-day workshop was to bring together international experts and young researchers in low-dimensional topology, number theory, string theory, quantum physics, algebraic geometry, conformal field theory, special functions and automorphic forms to discuss new developments and investigate potential directions for future research at the crossroads of modular forms and quantum knot invariants. A brief description of some future directions are as follows:

The Quantum Modularity and Volume conjectures: Quantum invariants of knots and 3-manifolds have been constructed rather combinatorially based on quantum groups, and their geometrical meanings are still unclear. The key is the Volume Conjecture, proposed by Kashaev and Murakami–Murakami, which states that, as $N \to \infty$, the N-th colored Jones polynomial $J_N(K;q)$ for a knot K at the N-th root of unity $q = e^{2\pi i/N}$ is dominated by the hyperbolic volume of the knot complement. The Volume Conjecture was proved for some knots such as the figure-eight knot and torus knots, but is still open for arbitrary K.

The relationship between quantum invariants and the hyperbolic volume motivated Zagier [21] to propose the notion of a "quantum modular form" as a function having nice properties at a root of unity. A typical example is the Kontsevich–Zagier series; it coincides, at a root of unity, with the Kashaev invariant for the trefoil, and furthermore can be written in terms of the Eichler integral of the Dedekind η-function. One goal would be to analyze the asymptotic behavior of the Kashaev invariants for hyperbolic knots and to clarify their quantum modularity. These are important problems from the viewpoint of quantum topology and modular forms.

Modularity of WRT invariants: A unified WRT invariant for integral homology spheres was recently proposed by Habiro. Interestingly, the computation of this unified WRT invariant for certain manifolds leads to new mock theta functions. For example, a result of Bringmann, Hikami and Lovejoy [3] says that a certain q-series $\phi(q)$ is a mock theta function and $\phi(-q^{1/2})$ is (up to an explicit factor) the unified WRT invariant of the Seifert manifold $\Sigma(2,3,8)$ which arises from $+2$ surgery on the trefoil knot.

One goal would be to study the unified WRT invariants of manifolds which arise from rational surgeries on other kinds of knots. These are typically quantum modular forms when viewed as functions at roots of unity, but when they converge inside the unit circle their behavior can vary. In some cases one obtains the usual indefinite binary theta functions related to mock theta functions, but in other cases one has less familiar expressions involving indefinite ternary quadratic forms or "partial" positive definite forms. An important task is to make the modularity properties of these unified WRT invariants explicit.

Stability: In 2006, Dasbach and Lin [6] observed stability in the coefficients of the colored Jones polynomial $J_N(K;q)$ for an alternating knot K. Precisely, they proved that the three leading coefficients of $J_N(K;q)$ are (up to a common sign) independent of N for $N \geq 3$. They then conjectured that for every N, the N leading coefficients of $J_N(K;q)$ are (up to sign) equal to the N leading coefficients of $J_{N+1}(K;q)$. This conjecture and its consequences have sparked a flurry of activity in both number theory and quantum topology. A few highlights are a resolution of this conjecture due to Armond [1] and (independently) Garoufalidis and Lê [9], the study of Rogers-Ramanujan type identities for "tails" of $J_N(K;q)$ from the perspectives of skein theory [12] and q-series [2], [16] and the study of potential stability of coefficients for generalizations of $J_N(K;q)$, namely for colored HOMFLY polynomials and colored superpolynomials.

Explicit methods for q-hypergeometric series: Quantum knot invariants are often expressible in terms of q-hypergeometric series, and explicit methods for these series are crucial. To give just two examples, the recent computation of the cyclotomic expansion of the colored Jones polynomial for the torus knots $(2,2t+1)$ depended on key advances in the method of Bailey pairs [13], while the proofs of conjectures of Garoufalidis, Lê and Zagier on the tails of alternating knots [16] relied heavily on the Andrews-Bowman generalization of Sears' identity. Further development of these techniques is still needed, especially with a view toward applications to properties of knot invariants.

References

[1] C. Armond, *The head and tail conjecture for alternating knots*, Algebr. Geom. Topol. **13** (2013), no. 5, 2809–2826.

[2] P. Beirne, R. Osburn, *q-series and tails of colored Jones polynomials*, Indag. Math. (N.S.) **28** (2017), no. 1, 247–260.

[3] K. Bringmann, K. Hikami and J. Lovejoy, *On the modularity of the unified WRT invariants of certain Seifert manifolds*, Adv. in Appl. Math. **46** (2011), no. 1-4, 86–93.

[4] K. Bringmann, A. Folsom, *Quantum Jacobi forms and finite evaluations of unimodal rank generating functions*, Arch. Math. (Basel) **107** (2016), no. 4, 367–378.

[5] K. Bringmann, J. Kaszian and A. Milas, *Vector-valued higher depth quantum modular forms and higher Mordell integrals*, preprint available at `https://arxiv.org/abs/1803.06261`

[6] O. Dasbach, X.-S. Lin, *On the head and the tail of the colored Jones polynomial*, Compos. Math. **142** (2006), no. 5, 1332–1342.

[7] R. Detcherry, E. Kalfagianni, *Quantum representations and monodromies of fibered links*, preprint available at `https://arxiv.org/abs/1711.03251`

[8] R. Detcherry, E. Kalfagianni, T. Yang, *Turaev-Viro invariants, colored Jones polynomials and volume*, preprint available at `https://arxiv.org/abs/1701.07818`

[9] S. Garoufalidis, T. Lê, *Nahm sums, stability and the colored Jones polynomial*, Res. Math. Sci. **2** (2015), Art. 1, 55pp.

[10] S. Garoufalidis, R. Kashaev, *A meromorphic extension of the 3D index*, preprint available at `https://arxiv.org/abs/1706.08132`

[11] S. Gukov, D. Pei, P. Putrov and C. Vafa, *BPS spectra and 3-manifold invariants*, preprint available at urlhttps://arxiv.org/abs/1701.06567

[12] M. Hajij, *The tail of a quantum spin network*, Ramanujan J. **40** (2016), no. 1, 135–176.

[13] K. Hikami, J. Lovejoy, *Torus knots and quantum modular forms*, Res. Math. Sci. **2** (2015), Art. 2, 15pp.

[14] K. Hikami, J. Lovejoy, *Hecke-type formulas for families of unified Witten-Reshetikhin-Turaev invariants*, Commun. Number Theory Phys. **11** (2017), no. 2, 249–272.

[15] S. Kanade, M. Russell, *Staircases to analytic sum-sides for many new integer partition identities of Rogers-Ramanujan type*, preprint available at `https://arxiv.org/abs/1803.02515`

[16] A. Keilthy, R. Osburn, *Rogers-Ramanujan type identities for alternating knots*, J. Number Theory **161** (2016), 255–280.

[17] R. Lawrence, D. Zagier, *Modular forms and quantum invariants of* 3*-manifolds*, Asian J. Math. **3** (1999), no. 1, 93–107.

[18] H. Murakami, *An introduction to the volume conjecture*, Interactions between hyperbolic geometry, quantum topology and number theory, 1–40, Contemp. Math., **541**, Amer. Math. Soc., Providence, RI, 2011.

[19] M. Stosic, P. Wedrich, *Rational links and DT invariants of quivers*, preprint available at `https://arxiv.org/abs/1711.03333`

[20] K. Walsh, *Higher order stability in the coefficients of the colored Jones polynomial*, preprint available at `https://arxiv.org/abs/1603.06957`

[21] D. Zagier, *Quantum modular forms*, Quanta of maths, 659–675, Clay Math. Proc., **11**, Amer. Math. Soc., Providence, RI, 2010.