Vertex Algebras and Quantum Groups

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1 Overview of the Field

During the twentieth century, the theory of Lie algebras, both finite- and infinite-dimensional, has been a major area of mathematical research with numerous applications. In particular, during the late 1970s and early 1980s, the representation theory of Kac-Moody Lie algebras (analogs of finite-dimensional semisimple Lie algebras) generated intense interest. The early development of the subject was driven by its remarkable connections with combinatorics, group theory, number theory, partial differential equations, and topology, as well as by its deep links with areas of physics, including integrable systems, statistical mechanics, and two-dimensional conformal field theory. In particular, the representation theory of an important class of infinite-dimensional Lie algebras known as *affine Lie algebras* led to the discovery of both vertex algebras and quantum groups in the mid-1980s. Motivated by the appearance of vertex operators in the construction of the Moonshine module, Borcherds introduced *vertex operator algebras* as part of his proof of the Moonshine Conjectures in 1986. (He received the Fields Medal for this accomplishment in 1998.) These are the precise algebraic counterparts to chiral algebras in 2-dimensional conformal field theory as formalized by Belavin, Polyakov, and Zamolodchikov. Among other applications, these algebras play a crucial role in string theory, integrable systems, M-theory, and quantum gravity. In 1985, the interaction of affine Lie algebras with integrable systems led Drinfeld and Jimbo to introduce *quantized universal enveloping algebras*, or *quantum groups*, associated with symmetrizable Kac-Moody algebras. These are q-deformations of the universal enveloping algebras of the corresponding Kac-Moody algebras, and like universal enveloping algebras, they carry an important Hopf algebra structure. The abstract theory of integrable representations of quantum groups, developed by Lusztig, illustrates the similarity between quantum groups and Kac-Moody Lie algebras. The theory of canonical bases for quantum groups has provided interesting insights to the representation theory of quantum groups. This has led to many applications in other areas of mathematics and mathematical physics.

Both vertex algebras and quantum groups are deeply intertwined with conformal field theory and integrable systems, though the exact nature of their relationship with each other remains somewhat mysterious. The main objective of the proposed meeting is to bring together experts in both of these domains in hopes of illuminating some of these connections.

2 Recent Developments

Motivated by developments in vertex algebras and quantum groups, Borcherds, Etingof-Kazhdan and E. Frenkel-Reshetikhin introduced certain deformations of vertex algebras called *quantum vertex* (or *chiral*) *algebras*. This promising idea was extended by Haisheng Li through several important publications. Interestingly, almost all of the most crucial questions about quantum vertex algebras remain unanswered. For instance, quantum affine algebras have been associated to weak quantum vertex algebras by Li in a certain conceptual way, but it still remains to construct the corresponding weak quantum vertex algebras explicitly and prove that they are quantum vertex algebras. Closely related to these developments, are recent breakthroughs in the area of quantum toroidal algebras, Yangians, quantum W-algebras and related structures. We expect that the proposed conference will provide an important and timely boost to research in this area.

It has been known since the early 1990s that rational vertex algebras coming from affine Lie algebras and quantum groups at roots of unity are closely related at the level of tensor categories. This connection often goes under the name of Kazhdan-Lusztig correspondence. Although the correspondence is functorial, it requires semisimplification at the level of quantum groups. Recently, motivated by a line of work in physics, it was conjectured that the category of modules for a certain irrational vertex algebra, called the triplet algebra, is equivalent to the representation category of the small quantum group associated to sl_2, without any semisimplification. The equivalence is expected to persist at the level of other vertex (super)algebras constructed via screening operators, called W-algebras, and small quantum groups in general. Moreover, the recent important work of Huang, Lepowsky and Zhang on logarithmic tensor product theory of vertex algebras provides a nice framework for studying these irrational vertex algebras from the categorical perspective. This workshop will report on progress in the area, as well as on connections with other areas of mathematics such as modular forms.

3 Presentation Highlights

Henning Haahr Andersen. *Tilting modules for quantum groups at roots of unity.*

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra denote by $U_v = U_v(\mathfrak{g})$ the associated quantized enveloping algebra over $\mathbb{Q}(v)$. We set $A = \mathbb{Z}[v, v^{-1}] \subset \mathbb{Q}(v)$ and let U_A denote the Lusztig A-form of U_v defined via quantum divided powers. Then for any field K and any parameter $q \in K$
$\{0\}$ we shall consider $U_q = U_A \otimes_A K$ (with K being an A-algebra via $v \mapsto q$) and its module category $\mathcal{C}_q$ consisting of all finite-dimensional U_q-modules (of type 1).

We identify $X = \mathbb{Z}^n$ ($n = \text{rk}\,\mathfrak{g}$) with the characters of U_q^0 (the "Cartan" subalgebra of U_q) and $X^+ = \mathbb{Z}^n_{\geq 0}$ with the set of dominant characters.

For each $\lambda \in X^+$ we have four modules in $\mathcal{C}_q$:
$\nabla_q(\lambda)$, the dual Weyl module
$L_q(\lambda) = \text{soc}\nabla_q(\lambda)$, the simple module
$\Delta_q(\lambda) = \nabla_q(\lambda)^*$, the Weyl module
$T_q(\lambda)$, the indecomposable tilting module,
all having λ as their unique highest weight. We have $\iota^\lambda : \ \Delta_q(\lambda) \hookrightarrow T_q(\lambda)$, $\pi^\lambda : \ T_q(\lambda) \twoheadrightarrow \nabla_q(\lambda)$ with $c^\lambda = \pi^\lambda \circ \iota^\lambda \neq 0$.

Questions: Q1. What is $\text{ch}L_q(\lambda)$?
Q2. What is $\text{ch}T_q(\lambda)$?

Remarks: a) If char$K = 0$ and ord $(q) = \ell$ then (with some slight conditions on ℓ) Q1 was solved in the early 1990's by Kazhdan and Lusztig plus Kashiwara and Tanisaki. Moreover, Q2 was solved a few years later by Soergel (with similar conditions on ℓ).
b) If char $K = p > 0$ and $q = 1$ then it is known that when $p \geq 2h - 2$ (h is the Coxeter number), then Q2 implies Q1. Actually, only a tiny (finite) part of Q2 is needed for this.
c) Very recently (December 2015) Riche and Williamson have (when char $K = p$ and $q = 1$) proposed a conjecture for Q2 and proved this conjecture for type A_n and $p > n$.

In an effort to understand the homomorphisms between tilting modules Stroppel, Tubbenhauer and I proved the following results (2015):

Theorem 1. Let $M, N \in \mathcal{C}_q$ and suppose M has a Δ_q-filtration and N has a ∇_q-filtration. If for each $\lambda \in X^+$ $\{f_j^\lambda | j = 1, \dots, m_\lambda\}$, resp. $\{g_i^\lambda | i = 1, \dots, n_\lambda\}$ is a basis for $Hom_{U_q}(Mn\nabla_q(\lambda))$, resp. $Hom_{U_q}(\Delta_q(\lambda), N)$, then the set $\{c_{ij}^\lambda = \overline{g}_i^\lambda \circ \overline{f}_j^\lambda | \lambda \in X^+, i, j\}$ is a basis for $Hom_{U_q}(M, N)$. Here $\overline{f}_j^\lambda$: $M \to T_q(\lambda)$, resp. $\overline{g}_i^\lambda : T_q(\lambda) \to N$ is any homomorphism such that $\pi^\lambda \circ \overline{f}_j^\lambda = f_j^\lambda$, resp. $\overline{g}_i^\lambda \circ \iota^\lambda = g_i^\lambda$.

Theorem 2. If T is a tilting module then $End_{U_q}(T)$ is a cellular algebra.

Proof: Take $T = M = N$ in Theorem 1. Then c_{ij}^λ is a cellular basis.

Example. If $\mathfrak{g} = sl(V)$ and V_q is the corresponding U_q-module for $U_q = U_q(sl(V))$ then $End_{U_q}(V_q^{\otimes^d})$ is cellular for all d.

Daniel Nakano. *Cohomology and support theory for quantum groups.*

Quantum groups are a fertile area for explicit computations of cohomology and support varieties because of the availability of geometric methods involving complex algebraic geometry. In this talk I will present results which illustrate these strong connections between the combinatorics, geometry and representation theory. We set the following notation.

- G: a simple, simply connected algebraic group over $\mathbb{C}$
- $\mathfrak{g} = \operatorname{Lie} G$
- h: Coxeter number
- ζ: primitive ℓth root of unity
- $\mathbb{U}_\zeta(\mathfrak{g})$: quantized enveloping algebra specialized at ζ
- $\mathbb{U}(\mathfrak{g})$: ordinary universal enveloping algebra
- $U_\zeta(\mathfrak{g})$: Lusztig $\mathcal{A}$-form specialized at ζ (distribution algebra)
- $u_\zeta(\mathfrak{g})$: small quantum group (f.d. Hopf algebra)
- $X = \mathbb{Z}\omega_1 \oplus \cdots \oplus \mathbb{Z}\omega_n$: the weight lattice, where $\omega_i \in \mathbb{E}$ are the fundamental weights.
- $X_+ = \mathbb{N}\omega_1 + \cdots + \mathbb{N}\omega_n$: the dominant weights
- $L_\zeta(\lambda)$: simple module $U_\zeta(\mathfrak{g})$ of highest weight $\lambda \in X_+$
- $\nabla_\zeta(\lambda) = \operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{g})} \lambda$: induced module where $\lambda \in X_+$
- $\Delta_\zeta(\lambda)$: Weyl module where $\lambda \in X_+$
- $T_\zeta(\lambda)$: tilting module of high weight $\lambda \in X_+$

In a fundamental computation, Ginzburg and Kumar demonstrated when $\ell > h$, the cohomology ring identifies with the coordinate algebra of the nilpotent cone of the underlying Lie algebra g=Lie(G).

Theorem. [Ginzburg-Kumar, 1993] Let $\mathfrak{g}$ be a complex simple Lie algebra and $\mathcal{N}$ be the nilpotent cone. If $l > h$ then

(a) $\mathrm{H}^{odd}(u_\zeta(\mathfrak{g}), \mathbb{C}) = 0$;

(b) $\mathrm{H}^{2\bullet}(u_\zeta(\mathfrak{g}), \mathbb{C}) = \mathbb{C}[\mathcal{N}]$.

When $\ell \leq h$, we calculated the cohomology ring and showed that the odd degree cohomology vanishes and in most cases $\mathrm{H}^{2\bullet}(u_\zeta(\mathfrak{g}), \mathbb{C}) = \mathbb{C}[G \cdot \mathfrak{u}_J]$ where $G \cdot \mathfrak{u}_J$ is the closure of some Richardson orbit (see [BNPP]). Our calculations heavily used the fact that

$$R^n \operatorname{ind}_{P_J}^G S^\bullet(\mathfrak{u}_J^*) = 0$$

for $n > 0$ (Grauert-Riemenschneider Vanishing Theorem). It is an open problem to prove this vanishing results over fields of characteristic $p > 0$.

With the finite generation of the cohomology ring one can define support varieties for quantum groups. Given $M \in \text{mod}(u_\zeta(\mathfrak{g})$, let $\mathcal{V}_{u_\zeta(\mathfrak{g})}(M)$ be the support variety of M. In general one can consider the "naive functor" which takes $M \in \text{mod}(U_\zeta(\mathfrak{g}))$ to G-equivariant coherent sheaves on the nilpotent cone (when $l > h$) by using the cohomology of the small quantum group. This functor was lifted to an equivalence of derived categories in the work of Arkhipov, Bezrukavnikov and Ginzburg (ABG).

Theorem. [Arkhipov-Bezrukavnikov-Ginzburg, 2003] Let $l > h$. There exists the following equivalences of derived categories

$$D^b(U_\zeta(\mathfrak{g})_0^{\text{mix}}) \cong D^b(Coh^{G\times\mathbb{C}^*}(G \times_B \mathfrak{n})) \cong D^b(Perv^{\text{mix}}(\text{Gr})).$$

Bezrukanikov used the ABG equivalence to compute the support varieties of tilting modules. For the induced/Weyl modules the support varieties were computed by Ostrik and Bendel-Nakano-Pillen-Parshall. The calculations for the simple modules employed the deep fact that the Lusztig Character Formula holds for quantum groups when $\ell > h$, and the positivity of the coefficients of the (parabolic) Kazhdan-Lusztig polynomial for the affine Weyl group. We present this in the following theorem.

Theorem. [Drupieski-Nakano-Parshall, 2012] Let $\ell > h$ and $\lambda \in X^+$. Choose $J \subseteq \Delta$ such that $w(\Phi_\lambda) = \Phi_J$ for some $w \in W$. Then

$$\mathcal{V}_{u_\zeta(\mathfrak{g})}(L_\zeta(\lambda)) = G \cdot \mathfrak{u}_J.$$

At the end of my talk a number of open problems described below were presented.

1) Calculate $\text{Ext}^n_{U_\zeta(\mathfrak{g})}(L(\lambda), L(\mu))$ for all $\lambda, \mu \in X_+$.

From the proof of the Lusztig Character Formula, this is known when λ and μ are regular weights. What about singular weights?

2) Is there a "rank variety" description of $\mathcal{V}_{u_\zeta(\mathfrak{g})}(M)$?

3) Let $\hat{\mathfrak{g}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}c \oplus \mathbb{C}d$ be the corresponding untwisted affine Lie algebra, and $\tilde{\mathfrak{g}}$ be the subalgebra $\tilde{\mathfrak{g}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}c \subseteq \hat{\mathfrak{g}}$. For $\kappa \in \mathbb{C}$ we let $\mathcal{O}_\kappa$ be the full subcategory of all $\tilde{\mathfrak{g}}$-modules, M, for which the central element c acts by κ and M satisfies category $\mathcal{O}$-type finiteness conditions. For a certain κ, there exists an equivalence of tensor categories

$$F_\ell : \mathcal{O}_\kappa \to \text{mod}(U_\zeta(\mathfrak{g}))$$

a) Is there any new information about support varieties and cohomology that can be gained by using this equivalence of categories?

b) How does the twisting of modules under the Frobenius morphism behave under this equivalence?

Evgeny Mukhin. *Trivial models with non-trivial Bethe Ansatz.*

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra with simple roots $\alpha_1, \ldots, \alpha_r$. Let L_λ be an irreducible $\mathfrak{g}$-module of highest weight λ. Consider $V = L_{\lambda_1} \otimes \ldots \otimes L_{\lambda_n}$, and let $(z_1, \ldots, z_n) \in \mathbb{C}^n$, $z_i \neq z_j$.

Define Gaudin Hamiltonians:

$$H_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\Omega^{(i,j)}}{z_i - z_j},$$

where $\Omega^{(i,j)}$ is the canonical element of $U\mathfrak{g} \otimes U\mathfrak{g}$ acting in i, j factors.

Example. For gl_{r+1}: $\Omega = \sum\limits_{i,j=1}^{r+1} e_{ij} \otimes e_{ji}$.

We have $[H_i, H_j] = 0$.

Problem. Find common eigenvectors and eigenvalues of H_i.

Since H_i commute with $U\mathfrak{g}$ diagonal action, we can restrict to V_λ^{sing}-subspace of V of singular vectors of weight λ. Let $\lambda = \lambda_1 + \ldots + \lambda_n - \sum_{i=1}^{r} \ell_i \alpha_i$, $\ell_i \in \mathbb{Z}_{\geq 0}$.

The problem is solved by Bethe Ansatz method. Let $t = (t_i^{(j)}\ , j = 1, \ldots, r, i = 1, \ldots, \ell_j)$ be auxiliary variables. One can explicitly define $W(t,z) \in V_\lambda$ called weight function and

$$\Phi = \prod \left(t_i^{(k)} - t_j^{(s)}\right)^{(\alpha_k,\alpha_s)} \prod \left(t_i^{(k)} - z_s\right)^{-(\alpha_k,\lambda_s)}.$$

Then one proves

Theorem. If

$$\left\{ \frac{\partial_{t_i^{(j)}} \Phi}{\Phi} = 0 \quad \text{for all } i, j, \right.$$

then $W(t,z)$ is an eigenvector of H_i and $W(t,z) \in V_\lambda^{sing}$.

What if $\dim(V_\lambda^{sing}) = 1$?

Message. Bethe Ansatz is still interesting!

Reasons:

- can be solved explicitly
- $t_i^{(j)}$ are zeros of interesting polynomials (such as Jacobi or Jacobi-Pi neiro polynomials)
- can be used to find sufficient amount of Bethe equation (3) in more general situation
- leads to explicitly computable multidimensional integrals (sl_2 gives the famous Selberg integrals)
- give meaningful recursions (???).

Fedor Malikov. *Strong homotopy chiral algebroids.*

Let A be $\mathbb{C}$-algebra. A Picard-Lie A-algebroid is an exact sequence

$$0 \longrightarrow A \longrightarrow L \longrightarrow T_A \longrightarrow 0,$$

where L is a Lie algebroid, i.e., an A-module and a Lie algebra, and the arrows preserve all the structure involved. The category of Picard-Lie A-algebroids is a torsor over the abelian group in categories $\Omega_A^{[1,2>}$. This result gives classification of algebras of twisted differential operators (TDO) over A, provided A is smooth.

If A is not smooth, then, following an idea of V. Hinich, we define an ∞-Picard-Lie algebroid, by using the exact sequence above but allowing the Lie algebras involved to be replaced with Lie$_\infty$-algebras. We prove that the arising groupoid is a torsor over $L\Omega_A^{[1,2>}$, the analogously truncated Illusie cotangent complex [Ill1, Ill2].

Our main result is that an analogous discussion allows to extend the result of [GMS] to obtain and classify ∞-chiral algebroids over a dga A. The definition of the latter is similar to the one just described except that the algebra A is replaced with the commutative chiral algebra $J_\infty A$, modules with chiral modules, and Lie algebras, such as T_A, with Lie*-algebras, such as $J_\infty T_A$, and then with Lie$^*_\infty$ algebras. The result is a groupoid that is a torsor over $L\Omega^{[1,2>}_{J_\infty A}(J_\infty T_A)$, where the latter is the homotopy version of the Chevalley-De Rham complex that arises in the Beilinson-Drinfeld compound pseudo-tensor category [BD].

Chongying Dong. *On orbifold theory.*

Orbifold theory studies a vertex operator algebra V under the action of a finite automorphism group G. The well-known Orbifold Theory Conjecture says that if V is rational then 1) V^G is rational, 2) every irreducible V^G-module occurs in an irreducible g-twisted V-module for some $g \in G$.

The appearance of the twisted modules is the main feature of the orbifold theory. In the case that V is the vertex operator algebra associated to the highest weight module for an affine Kac-Moody algebra, the g-twisted modules are exactly the modules for the twisted affine Kac-Moody algebras.

Here are our main results: Assume 1) V is a rational vertex operator algebra and G is a finite automorphism group of V, 2) the weight of every irreducible g-twisted module ($g \in G$) is positive except for V itself. Then every irreducible V^G-module occurs in an irreducible g-twisted V-module for some $g \in G$.

This result reduces the orbifold theory conjecture to the rationality of V^G. The main tool we use to prove the result is the modular invariance of trace functions by Zhu, Dong-Li-Mason. The assumption that any irreducible g-twisted V-module has a positive weight except for V itself holds for the most well known rational vertex operator algebras. This assumption allows us to use the tensor product and related results given by Huang and Dong-Li-Ng.

We also find an interesting formula on the global dimension of V. The global dimension is defined to be the sum of squares of the quantum dimensions of inequivalent irreducible V-modules. It turns out the global dimension of V is a sum of the squares of the quantum dimensions of inequivalent irreducible g-twisted V-modules for any finite order automorphism g.

Here are some important open problems in the orbifold theory:

Conjecture 1. If V is rational and G is a finite automorphism group of V then V^G is rational.

Conjecture 2 . If V is rational then V is g-rational for any finite order automorphism g.

Conjecture 3. If V is rational and G is a finite automorphism group of V, then there exists a finite-dimensional semi-simple Hopf algebra A such that the module category of V^G and the module category of A are equivalent as tensor categories.

Terry Gannon. (joint with T. Creutzig) *The theory of C_2-cofinite VOAs.*

The theory of rational VOAs is quite well understood.

1. The corresponding category of modules has finitely many simples, has direct sums, is semisimple, has tensor product, is rigid. Such a category is called a *fusion category*. The category of finite-dimensional representations over $\mathbb{C}$ of a finite group is a prototypical example of a fusion category. The category of rational VOA is even better: it is a *modular tensor category* (MTC). This means it is a fusion category which is ribbon and satisfies $Z(\mathcal{C}) \cong \mathcal{C} \boxtimes \mathcal{C}^{op}$, where $Z(\mathcal{C})$ is Drinfeld double. A modular tensor category has finite-dimensional representations of all surface mapping class groups, e.g. $SL_2(\mathcal{Z})$.

Conjecture. Every MTC is a category of modules of a rational VOA.

Fusion categories can be studied by subfactor methods, and we find that we seem to know perhaps 0% of all fusion categories. Taking their doubles give MTCs, which conjecturally correspond to VOAs. So it is tempting to guess that we know 0% of rational VOAs. In other words, we are missing general construction.

2. Zhu proved that the character $Z_\mu(\tau) = \mathrm{Tr}_\mu q^{L_0 - c/24}$ of rational VOAs form a vector-valued modular form of weight 0 for $SL_2(\mathbb{Z})$. There is a generalization, where $u \in V$ are inserted, giving vector-vlaued modular forms of higher weight.

3. Verlinde's formula expresses the tensor product coefficients $M_i \otimes M_j \cong \bigoplus_k N_{ij}^k M_k$ in terms of the matrix $S = \rho \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ coming for the $SL_2(\mathbb{Z})$-multiple in # 2:

$$N_{ij}^k = \sum_\ell \frac{S_{i\ell} S_{j\ell} \overline{S}_{k\ell}}{S_{1\ell}}.$$

The "elementary" part of this is that Verlinde's formula holds for S associated to the Hopf link in the MTC (this was proved by Turaev). The deep part is that S also appears in # 2.

If we drop semisimplicity for the definition of a rational VOA, we get the class of C_2-cofonite VOAs. These should be a gentle generalization of rational VOAs. We should expect that almost all C_2-cofinite VOAs are not rational. But there are only 2 families that we studied: W_p triplet models, and symplectic fermions SF_d. More families of examples are needed!

1. Huang / Miyamoto proved that the category of C_2-cofinite VOAs have finitely many simples, each has projective cover, all Hom spaces are finite, every module has finite composition series, it is an abelian tensor category. Missing: rigidity. Apart from rigidity, it is a *finite tensor category*, the nonsemisimple version of fusion category. Also it is braided. Tempting to guess that nonsemisimple version of MTC is a ribbon finite tensor category satisfying $Z(\mathcal{C}) \cong \mathcal{C} \boxtimes \mathcal{C}^{op}$. A big question is to relate this to Lyubashenko's work, which builds representations of mapping class groups from a closely related category.

Theorem. If V is C_2-cofinite, rigid, not rational, then V has uncountably many indecomposables for infinitely many composition series lengths.

For example, W_p has 4 families of indecomposables for each composition length, each parametrized by $\mathbb{CP}^1$ (tame-type representations). SF_d for $d > 1$ has wild representation type.

Expectation: almost every C_2-cofinite VOA has wild representation type – their indecomposables will never be classified.

Subfactor methods should extend to finite tensor categories, but may not be effective.

2. The characters of C_2-cofinite VOAs can be defined, but look like weight-0 piece + weight-1 piece $+\ldots+$ weight-N piece. When $SL_2(\mathbb{Z})$ is applied to them, the weight k piece picks up τ^k. So $\mathbb{C}$-span of characters are not closed under $SL_2(\mathbb{Z})$. To get a vector-valued modular form, we have to define new pseudo-characters with τ, q exponents.

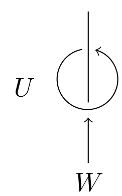
Physics approach (e.g. Runkel): Insert intertwining vertex operator of type $\binom{M}{P_V, M}$, where P_V is projective cover. At least for symplectic fermions, this gives missing pseudocharacters.

Associative algebra approach (Miyamoto): use symmetric linear functionals, etc. This always works but is not very effective. Might be overkill: Arike-Nagatomo showed for symplectic fermions it gives dimension $\geq 2^{2d-1}+3$, but the modular completion of characters is $d+4$-dimensional.

Big question: Can physics approach be reconciled with Miyamoto's? Is Miyamoto's overkill?

3. Search for categorical analogue of Verlinde. Block diagonalise fusion matrices for Grothendieck ring: e.g. for W_p get 2 1×1 blocks, $p-1$ 2×2 blocks; for SF_d get 2 1×1 blocks, 1 2×2 blocks. These blocks coincide with subrep of open Hopf links: let P be projective, $\Phi_{w,p} \in Z(End(P))$. $\Phi_{w,p}$ define representation of fusion ring, subreps are those blocks.

$\Phi_{U,W} \quad =$



Tempting to guess this always happens for C_2 cofinite VOAs. No analogue of this for full tensor product ring.

Dražen Adamović. *Conformal embeddings and realizations of certain simple W-algebras.*

In this talk we present certain new results and constructions related with conformal embeddings, extension theory of vertex algebras, and with W-algebras appearing in logarithmic conformal field theory.

First we discuss notion of simple current modules for affine VOA $V_k(\mathfrak{g})$ in the category KL_k. We present a result (D. Adamović, O. Perse [AP]) on new simple current modules for $V_{-1}(sl_n)$. Then motivated by a conformal embedding $V_k(gl_n)$ into $V_k(sl_{n+1})$, for $k=-\frac{n+1}{2}$ [AKMPP1], we present a new conjectural list of simple current modules.

More precisely, for $i\in\mathbb{Z}$, let $M_{k,i} = L_{sl(n)}(\lambda_{k,i})$ where $\lambda_{k,i} = (k-i)\Lambda_0 + i\Lambda_1$ $(i\geq 0)$, $\lambda_{k,i} = (k+i)\Lambda_0 - i\Lambda_n$ $(i<0)$.

Conjecture Let $k=-\frac{n+1}{2}$ $(n\geq 4)$. In the category KL_k of $V_k(sl(n))$–modules, the following fusion rules holds: $M_{k,i}\times M_{k,j} = M_{k,i+j}$.

Next we present new results on conformal embeddings of affine vertex algebras into W-algebras:

Theorem[AKMPP2]. Let $k=-\frac{2}{3}(n+2)$. Then we have conformal embedding $V_{k+1}(gl(n))$ in $W_k = W_k(sl(n+2), f_\theta)$. Assume that $n\geq 3$. Then $W_k = \bigoplus_{i\in\mathbb{Z}} W_k^{(i)}$ and each $W_k^{(i)}$ is irreducible $V_{k+1}(gl(n))$–module.

We show that results on conformal embeddings can imply new construction of extension of affine VOAs. A particular emphasis is put on the conformal embedding $V_{k+1}(gl_n)$ in $W_k(sl_{n+2}, f_\theta)$, for $k=-\frac{2}{3}(n+2)$.

A complete reducibility result for this conformal embedding is presented.

Next we show that vertex algebras $V_{-\frac{3}{2}}(sl_3)$ and $W_{-\frac{8}{3}}(sl_4, f_\theta)$ are related with vertex algebras appearing in logarithmic conformal field theory. Explicit realization of $W_{-\frac{8}{3}}(sl_4, f_\theta)$ from [AKMPP2] was discussed. We also present a result from [A] that parafermionic vertex algebra $K(sl_3, -\frac{3}{2})$ is isomorphic to the W–algebra $W^0_{A_2}(p)$, for $p=2$, where $W^0_{A_2}(p)$ is a higher rank generalization of the triplet vertex algebra $W(p)$ from [AM]. We discuss a conjecture that $W^0_{A_2}(p)$ is a C_2–cofinite, non-rational vertex algebra.

Valerio Toledano Laredo. *Yangians, quantum loop algebras and elliptic quantum groups.*

The Yangian $Y\mathfrak{g}$ and quantum loop algebra $U_q(L\mathfrak{g})$ of a complex semisimple Lie algebra $\mathfrak{g}$ share very many similarities, and were long thought to have the same representations, though no precise relation between them existed until recently.

I will explain how to construct a faithful functor from the finite-dimensional representations of $Y\mathfrak{g}$ to those of $U_q(L\mathfrak{g})$. The functor is entirely explicit, and governed by the monodromy of the abelian difference equations determined by the commuting fields of the Yangian. It yields a meromorphic, braided Kazhdan-Lusztig equivalence between finite-dimensional representations of the $Y\mathfrak{g}$ and of $U_q(L\mathfrak{g})$.

A similar construction yields a faithful functor from representations of $U_q(L\mathfrak{g})$ to those of the elliptic quantum group $E_{q,\tau}(\mathfrak{g})$ corresponding to $\mathfrak{g}$. This allows in particular a classification of irreducible finite-dimensional representations of $E_{q,\tau}(\mathfrak{g})$, which was previously unknown.

This is joint work with Sachin Gautam (Perimeter Institute).

Naihuan Jing (joint work with N. Rozhkovskaya). *Vertex operators and Giambelli identities.*

1. Giambelli identity.

Classical Jacobi-Trudi identity says that for any partition $\lambda = \lambda_1 + \ldots + \lambda_\ell$ ($\lambda_1 \geq \ldots \geq \lambda_\ell > 0$) the Schur function s_λ is the determinant of certain homogeneous symmetric function $h_n := s_{(n)}$:

$$s_\lambda = det(h_{\lambda_i - i + j})_{\ell \times \ell}.$$

In a series of classical papers of Weyl in 1920's, he gave similar determinant formula for the character of simple modules of classical Lie simple algebras (in types B, C, D), e.g.,

$$sp_\lambda = \frac{1}{2} det(h_{\lambda_i - i + j} + h_{\lambda_i - i - j + 2}),$$

where h_n is the character of simple module corresponding to $\square\square\cdots\square$.

Our second motivation comes from Okounkov-Olshanski formula for the shifted Schur function

$$s^*_\lambda = \frac{det(x_i + n - i | \lambda_j + n - j)}{det(x_i + n - i | n - j)},$$

where $(x|k) = x(x-1)\ldots(x-k+1)$ for $k \geq 1$; and $\frac{1}{(x+1)\ldots(x+(-k))}$ for $k < 0$.

We introduce the so-called generalized Schur function s_λ (still the same notation) for any composition $\lambda = \lambda_1 + \ldots + \lambda_\ell$ in the ring $B = \mathbb{C}[h_1^{(0)}, h_2^{(0)}, \ldots]$ and $B^{\leq n}$ = subspace of deg $\leq n$. In the general generators $h_k^{(r)} \in B^{\leq r+k}$, $h_{-k}^{(k)} = 1$ and $h_k^{(r)} = 0$ for $k + r < 0$. We define

$$s_\lambda \doteqdot det\left[h^{(j-1)}_{\lambda_i - i + 1}\right].$$

Then we can define the generalized elementary symmetric function $e_a^{(p)}$ and the essential part is

$$e_a^{(p)} = det\left[h^{(-p+j)}_{p+1-i}\right], p > a,$$

and other trivial cases for $p \leq a$. Then one can introduce the hook Schur function

$$s_{(m|n)} = \sum_{p=0}^{n} (-1)^p h^{(p)}_{m+1} e^{(-p)}_{-n}.$$

Proposition (Newton-like formula).

$$\sum_{p=-\infty}^{\infty} (-1)^{a-p} h_b^{(p)} e_a^{(-p)} = \delta_{ab}, \ a, b \in \mathbb{Z}.$$

Theorem (Giambelli-like). For every partition $(\alpha|\beta) = (\alpha_1 \ldots \alpha_r | \beta_1 \ldots \beta_r)$ in Frobenius notation of a partition, and every $n \geq \ell(\alpha|\beta)$,

$$s_{(\alpha|\beta)} = det\left[s_{(\lambda_i - i | n - j)}\right]_{n \times n} = det\left[s_{(\alpha_i | \beta_j)}\right]_{r \times r}.$$

2. Vertex operator realization of generalized Schur functions.
Define the action of ψ_k, ψ_k^* on $B = \underset{m\in\mathbb{Z}}{\oplus} B^{(m)}$, $B^{(m)} = z^m\mathbb{C}[s_\lambda]$:

$$\psi_k(s_\lambda z^m) = s_{(k-m-1,\lambda)} z^{m+1},$$

$$\psi_k^*(s_\lambda z^m) = \sum_{t=1}^{\infty} (-1)^{t+1}\delta_{k-m-1,\lambda_t-t} s(\lambda_1+1,\ldots,\lambda_t+1,\lambda_{t+1},\ldots)z^{m-1}.$$

Theorem. 1. The operators ψ_k, ψ_k^* satisfy the Clifford algebra relations

$$\{\psi_k, \psi_\ell\} = \{\psi_k^*, \psi_\ell^*\} = 0,\ \{\psi_k, \psi_\ell^*\} = \delta_{k,\ell}.$$

2. For any partition λ, with $\mu = \lambda'$, the dual of λ,

$$\psi_{\lambda_\ell+\ell}\ldots\psi_{\lambda_1+1}\dot{1} = s_\lambda z^\ell,$$

$$\psi^*_{-\lambda_1-\ell+1}\cdots\psi^*_{-\lambda_\ell}\dot{1} = (-1)^{|\lambda|}s_\mu z^{-\ell}.$$

(which imply that the module of the Clifford algebra is simple).

We then can write down the vertex operator realization of the generalized Schur functions.

Haisheng Li. *q-Virasoro algebra and affine Lie algebra.*

This talk is based on a joint work with H. Guo, S. Tan and Q. Wang. The so-called q-Virasoro algebra is an infinite-dimensional Lie algebra introduced by Belov and Chaltikian many years ago. In this talk, I will first review a theory of equivariant quasi modules for vertex algebras, which was developed by the author. Then I will explain how one can associate vertex algebras to the q-Virasoro algebra and show that the q-Virasoro with q a root of unity of order $2\ell+1$ is actually isomorphic to the affine Kac-Moody Lie algebra of type $B_\ell^{(1)}$.

First, we review the theory of quasi modules for vertex algebras. Let W be an arbitrary vector space over $\mathbb{C}$. Set $\mathcal{E}(W) = Hom(W, W((x)))$. A subset U of $\mathcal{E}(W)$ is said to be quasi-local if for any $a(x), b(x) \in U$ there exists a non-zero polynomial $p(x,z)$ such that

$$p(x,z)a(x)b(z) = p(x,z)b(z)a(x).$$

Assume $a(x), b(x) \in U$ are quasi-local. Define $a(x)_n b(x) \in \mathcal{E}(W)$ for $n \in \mathbb{Z}$ in terms of generating function

$$Y_{\mathcal{E}}(a(x), z)b(x) = p(x+z, z)\left(p(x_1, x)a(x_1)b(x)\right)_{x_1=x+z}.$$

Theorem. Every quasi-local subset U of $\mathcal{E}(W)$ generates a vertex algebra $\mathcal{W}$ with W as a quasi module.

Now consider the q-Virasoro algebra $\mathcal{D}$, which is a Lie algebra with generators $D^\alpha(n)$ for $\alpha, n \in \mathbb{Z}$ and C, satisfying certain relations. For each$\alpha \in \mathbb{Z}$, form a generating function

$$D^\alpha(x) = \sum_{n\in\mathbb{Z}} D^\alpha(n)x^{-n-1}.$$

The fact is that these generating functions are quasi-local.

Theorem. [Guo-Li-Tan-Wang] $\mathcal{D}$ is isomorphic to the covariant Lie algebra $\widehat{\mathfrak{g}}/S$ of the affine Lie algebra $\widehat{\mathfrak{g}}$, where $\mathfrak{g}$ is a certain Lie algebra with an automorphism group S

Theorem. [Guo-Li-Tan-Wang] The category of restricted $\mathcal{D}$-modules of level ℓ is isomorphic to the category of equivariant quasimodules for the vertex algebra $V_{\widehat{\mathfrak{g}}}(\ell, 0)$.

Theorem. [Guo-Li-Tan-Wang] Assume that q is a root of unity of order $2\ell+1$. Then $\mathcal{D}$ is isomorphic to the affine Kac-Moody algebra of type $B_\ell^{(1)}$.

Iana I. Anguelova. *Towards quantum chiral algebras.*

In 1988 I. Frenkel and N. Jing constructed the vertex representations of the quantum affine algebras, starting with the quantum Heisenberg operators $a_{ik}, k \in \mathbb{Z}$ satisfying the relations ($\gamma = e^{tc/2} = q^c$)

$$[a_i(m), a_i(n)] = \delta_{i,-m}\frac{1}{mt^2}(q^{mA_{ij}} - q^{-mA_{ij}})(q^m - q^{-m}),$$

and defining the vertex operators

$$Y_i^{\pm}(z) = exp\Big(\pm t\sum_{n\geq 1}\frac{q^{\mp\frac{n}{2}}}{q^n-q^{-n}}a_i(-n)z^n\Big)\times exp\Big(\mp t\sum_{n\geq 1}\frac{q^{\mp\frac{n}{2}}}{q^n-q^{-n}}a_i(n)z^{-n}\Big)a_i^{\pm 1}z^{\pm a_i(0)+1}.$$

These vertex operators (together with $\Phi_i(z)$ and $\Psi_i(z)$) define a representation of the quantum affine algebras at $c = 1$.

The fundamental problem posed by Igor Frenkel then is to formulate and develop a suitable theory of quantum vertex algebras (quantum chiral algebras) incorporating as examples the Frenkel-Jing quantum vertex operators. As one works towards an answer to this problem one of the big questions that needs to be addressed at the start is: do we require Completeness with respect to Operator Product Expansions (OPEs) for our quantum chiral algebras? I.e., do we require that the coefficients in the OPEs are vertex operators in the **same** (quantum) vertex algebra? Here, as in physics, the term "*chiral algebra*" will refer to the fact that we do require completeness with respect to the OPEs. For example, in the quantum affine case (FJ) we have OPEs such as:

$$Y_i^{\mp}(z)Y_i^{\pm}(w)\sim\frac{q^2}{q^2-1}\frac{w:Y_i^{\pm}(qw)Y_i^{\mp}(w):}{z-qw}-\frac{w:Y_i^{\pm}(q^{-1}w)Y_i^{\mp}(w):}{(q^2-1)(z-q^{-1}w)}$$

This OPE reflects the fact that for the quantum affine algebras we have poles in the OPEs not only at $z = w$, but in fact at $z = q^n w$, $n \in \mathbb{Z}$ (here $q \in \mathbb{C}$). We observe that both in this case (trigonometric) and in the elliptic case (such as the deformed Virasoro chiral algebra of E. Frenkel and N. Reshetikhin), it is clear that the OPE completeness requires field descendants of the type $\mathbf{a}(\gamma\mathbf{z})$ for $\gamma \neq 1$. This has profound consequences: in particular if both the fields $a(z)$ and $a(\gamma z)$ are to be incorporated in the **same chiral algebra structure**, this would result in the state-field correspondence becoming non-invertible! Recall the **state-field correspondence** is a map from the space of states W to the space of fields V, given by $W \ni a \mapsto a(z) = Y(a, z) \in V$ (bijection for super vertex algebras). Its inverse map is the **field-state correspondence**, a map from the space of fields V to the space of states W on which the fields act, defined by $V \ni \tilde{a}(z) \mapsto a := \tilde{a}(z)|0\rangle|_{z=0} \in W$. If both the fields $a(z)$ and $a(\gamma z)$ have to be incorporated in the same chiral algebra , then the field-state correspondence map will send the different fields $a(z)$ and $a(\gamma z)$ to the same state element $a \in W$:

$$a(\gamma z)|0\rangle|_{z=0} = a(z)|0\rangle|_{z=0} = a \in W.$$

Thus the space of fields V will be a (ramified) cover of the space of states W. Our proposed definition of a chiral algebra with Γ-type singularities reflects this. A particular "baby" class of chiral algebras with Γ-type singularities is the class where the group of singularities is the roots of unity group $\Gamma = \{1, \epsilon, \dots, \epsilon^{N-1}\}$ and the R-matrix is $R(a \otimes b) = (-1)^{p(a)p(b)}a \otimes b$ (still trivial). We called this particular subclass of chiral algebras "twisted vertex algebras" (IA). They represent the simplest nontrivial examples of chiral algebras with OPE singularities located at points besides $z = w$. Nevertheless they are interesting in its own accord as they reflect the cases of the boson-fermion correspondences of types B, C and D, which are all isomorphisms of twisted vertex algebras. The details on the bosonizations of types B, C and D can be found in a series of my papers, which have delineated these correspondences and some of their properties.

In conclusion, we would like to thank the organizers and BIRS for the extremely productive Workshop, and we wanted to remark that we truly appreciated the great atmosphere at the Banff Centre.

Andrew Linshaw. *Orbifolds and cosets via invariant theory.*

Given a vertex algebra $\mathcal{V}$ and a group of automorphisms $G \subset \text{Aut}(\mathcal{V})$, the invariant subalgebra $\mathcal{V}^G$ is called an *orbifold* of $\mathcal{V}$. Similarly, given a vertex subalgebra $\mathcal{A} \subset \mathcal{V}$, the subalgebra $\text{Com}(\mathcal{A}, \mathcal{V}) \subset \mathcal{V}$ which commutes with $\mathcal{A}$ is called a *coset* of $\mathcal{V}$. Many interesting vertex algebras can be constructed either as orbifolds or cosets, or as extensions of these structures. It is widely believed that nice properties of $\mathcal{V}$ such as strong finite generation, C_2-cofiniteness, and rationality, will be inherited by $\mathcal{V}^G$ and $\text{Com}(\mathcal{A}, \mathcal{V})$ if G and $\mathcal{A}$ are also nice, but so far few general results of this kind are known.

In recent work, we have established the strong finite generation of orbifolds of free field algebras under arbitrary reductive automorphism groups. This should be viewed as a vertex algebra analogue of Hilbert's theorem that if a reductive group G acts on a finite-dimensional complex vector space V, the ring $\mathbb{C}[V]^G$ of invariant polynomial functions is finitely generated. The proof of our result makes use of the structure and representation theory of $\mathcal{V}^{\mathrm{Aut}(\mathcal{V})}$ when $\mathcal{V}$ is a free field algebra, together with the fact that all $\mathcal{V}^{\mathrm{Aut}(\mathcal{V})}$-modules appearing in the decomposition of $\mathcal{V}$ have the C_1-cofiniteness property according to Miyamoto's definition.

In joint work with T. Creutzig, we have shown that a broad class of cosets of affine vertex algebras inside larger vertex algebras can be described by passing to a certain limit, which is an orbifold of a free field algebra. These cosets depend continuously on the central charge, and a strong generating set for the limit gives rise to a strong generating set for the coset at generic points in the family. In this way, we have established the strong finite generation of a broad class of cosets, and in many instances we can give an explicit minimal strong generating set.

In this conference we discussed these results, as well as recent joint work with T. Arakawa, T. Creutzig, and K. Kawasetsu in which we extended our methods to orbifolds and cosets of the minimal $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g}, f_{\mathrm{min}})$ associated to a simple Lie algebra $\mathfrak{g}$ and its minimal nilpotent element f_{min}. For example, we obtained the following description of the coset of the affine vertex algebra generated in weight one inside $\mathcal{W}^k(\mathfrak{g}, f_{\mathrm{min}})$, when $\mathfrak{g}$ is either $\mathfrak{sl}_n$ or $\mathfrak{sp}_{2n}$.

Theorem. For $n \geq 3$ and for generic values of k, the coset $\mathcal{C}^k = \mathrm{Com}(V^{k+1}(\mathfrak{gl}_{n-2}), \mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{min}}))$ is of type $\mathcal{W}(2, 3, \dots, n^2 - 2)$. In other words, a minimal strong generating set consists of a field in each weight $2, 3, \dots, n^2 - 2$.

Theorem. For $n \geq 2$ and for generic values of k, the coset $\mathcal{C}^k = \mathrm{Com}(V^{k+1/2}(\mathfrak{sp}_{2n-2}), \mathcal{W}^k(\mathfrak{sp}_{2n}, f_{\mathrm{min}}))$ is of type $\mathcal{W}(2, 4, \dots, 2n^2 + 4n)$.

We also discussed some cases where the set of nongeneric values of the central charge (where our description of the coset does not work) can be described explicitly. This allowed us to describe the coset of the *simple* affine vertex algebra inside the simple minimal $\mathcal{W}$-algebras in several examples.

Nicolas Guay. *Twisted Yangians for symmetric pairs of types B,C,D and their representations.*

This is joint work in progress with Vidas Regelskis and Curtis Wendlandt.

Yangians are one of the important families of affine quantum groups and they afford a rich representation theory. Theoretical physics is a source of motivation for their study, one reason being that representations of Yangians lead to solutions of the quantum Yang-Baxter equation, another one being that Yangians control certain symmetries of integrable systems. For integrable systems with boundaries, it turns out that so-called twisted Yangians are sometimes relevant for the study of their symmetries. In the mathematical literature, the twisted Yangians that have been the most studied were introduced by G. Olshanski about twenty-five years ago and are associated to the symmetric pairs $(\mathfrak{gl}_n, \mathfrak{so}_n)$ and $(\mathfrak{gl}_n, \mathfrak{sp}_n)$. Over the years, their representation theory has been investigated in several papers of S. Khoroshkin, A. Molev, M. Nazarov, V. Toledano Laredo and others. Our ongoing project is to introduce similar twisted Yangians for other classical symmetric pairs, study their properties and develop their representation theory.

Given a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ and an involution ρ of $\mathfrak{g}$, the pair $(\mathfrak{g}, \mathfrak{g}^\rho)$ is called a symmetric pair. We are interested mainly in the cases when $\mathfrak{g}$ is $\mathfrak{so}_N$ or $\mathfrak{sp}_N$ and $\mathfrak{g}^\rho$ is isomorphic to $\mathfrak{gl}_{\frac{N}{2}}$ (if N is even) or $\mathfrak{so}_p \oplus \mathfrak{so}_{N-p}$ or $\mathfrak{sp}_p \oplus \mathfrak{sp}_{N-p}$ for some $0 \leq p \leq N$. The involution ρ can be extended to the current algebra $\mathfrak{g} \otimes \mathbb{C}[t]$ via $\rho(X \otimes t^k) = \rho(X) \otimes (-t)^k$ and the invariant subalgebra $\mathfrak{g}[t]^\rho$ is called a twisted current algebra.

The twisted Yangian $Y^{tw}(\mathfrak{g}, \mathfrak{g}^\rho)$ is a deformation of the enveloping algebra of $\mathfrak{g}[t]^\rho$. It can be defined as a coideal subalgebra of the Yangian $Y(\mathfrak{g})$ of $\mathfrak{g}$ via the RTT-presentation of $Y(\mathfrak{g})$. Actually, we first obtain the extended twisted Yangian $X^{tw}(\mathfrak{g}, \mathfrak{g}^\rho)$ as a coideal subalgebra of the extended Yangian $X(\mathfrak{g})$ of $\mathfrak{g}$ and $Y^{tw}(\mathfrak{g}, \mathfrak{g}^\rho)$ can be obtained either as a subalgebra of $X^{tw}(\mathfrak{g}, \mathfrak{g}^\rho)$ or as a quotient by an ideal generated by central elements. We prove that, equivalently, $X^{tw}(\mathfrak{g}, \mathfrak{g}^\rho)$ can be defined by generators satisfying the reflection equation and a symmetry relation. These basic results parallel those for Olshanski's twisted Yangians and we are able to recover these when $\mathfrak{g} = \mathfrak{so}_3 \cong \mathfrak{sl}_2$ or $\mathfrak{g} = \mathfrak{sp}_2 = \mathfrak{sl}_2$.

A natural first step in the study of representations of twisted Yangians is to classify their irreducible finite dimensional modules. For $Y(\mathfrak{g})$, these modules are classified by certain tuples of polynomials. This is still true for twisted Yangians. To obtain a classification theorem, we start by proving that finite dimensional irreducible modules are highest weight modules, hence are quotients of Verma modules. It is thus sufficient to determine when such a module is non-trivial and when its non-zero irreducible quotient is finite dimensional.

The proof of the criterion for finite dimensionality can be split into two parts: the first one consists of using induction via a reduction on the rank of $\mathfrak{g}$; the second step is to give a direct proof when the rank of $\mathfrak{g}$ is minimal. In some cases, this second part follows immediately from know results for the twisted Yangians of $\mathfrak{sl}_n$.

Kiyokazu Nagatomo. *Vertex operator algebras, minimal models and modular linear differential equations.*

On this occasion I discuss two ways to characterize the minimal series of Virasoro vertex operator algebras L_c with the central charge c and 4 simple modules. Let V be a VOA. We propose two characterizations which are about characters of V-modules. More explicitly one is that the second and third coefficients of the character of V are 0 and 1, respectively, and other is that the second coefficients of characters of V-modules except V are all 1. Then we show that the first condition confines to $L_{-46/3}$, $L_{-3/5}$ and the two VOAs which are extension of $L_{-114/7}$ and $L_{4/5}$ by their simple modules. The second conditions also gives $L_{-46/3}$, $L_{-3/5}$, and the lattice VOA V_L with $L = \mathbb{Z}\alpha$ and $\langle \alpha, \alpha \rangle = 6$.

Vidas Regelskis. *Classification of trigonometric reflection matrices.*

This is a joint work with Bart Vlaar. Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra and let $\theta : \mathfrak{g} \to \mathfrak{g}$ be an involutive Lie algebra automorphism. Let $\mathfrak{k} = \{x \in \mathfrak{g} : \theta(x) = x\}$ denote the theta fixed subalgebra. Assuming that θ is of a second kind, that is $\dim(\theta(\mathfrak{b}^+) \cap \mathfrak{b}^+) < \infty$, where $\mathfrak{b}^+$ is the standard Borel subalgebra, there exists a quantum group analogue $B^{\mathbf{c},\mathbf{s}} = B^{\mathbf{c},\mathbf{s}}(\theta)$ of the universal enveloping algebra $U(\mathfrak{k})$, which is a left coideal subalgebra of the Drinfeld-Jimbo quantized universal enveloping algebra $U_q(\mathfrak{g})$. In a suitable setting, the algebras $B^{\mathbf{c},\mathbf{s}}$ are in a bijection with Satake diagrams. In his talk, we focused on the case when $\mathfrak{g}$ is an untwisted affine Lie algebra of classical type: $\widehat{\mathfrak{sl}}_N$, $\widehat{\mathfrak{so}}_N$ or $\widehat{\mathfrak{sp}}_N$. In this setting the natural (vector) representation of $U_q(\mathfrak{g})$ on the N-dimensional vector space is also an irreducible representation of its coideal subalgebras $B^{\mathbf{c},\mathbf{s}}$ indexed by affine Satake diagrams. We explain a method to obtain solutions of the Sklyanin's twisted and untwisted reflection equations. In this approach reflection matrices are defined as the intertwiners of the natural representation of $U_q(\mathfrak{g})$ restricted to a given subalgebra $B^{\mathbf{c},\mathbf{s}}$. In particular, he described a class of so-called quasistandard reflection matrices, that have elegant representation-theoretic properties.

Simon Wood. *The rationality of $N = 1$ minimal models through symmetric polynomials.*

Let $V(c)$ be the universal $N = 1$ Virasoro vertex algebra at central charge $c \in \mathbb{C}$. Unless

$$c = c_{p_+,p_-} = \frac{3}{2} - 3\frac{(p_+ - p_-)^2}{p_+p_-}\,,$$

for some $p_+, p_- \geq 2$, $\gcd\{\frac{p_- - p_+}{2}, p_+\}$, the vertex algebra $V(c)$ is simple as a module over itself. However, $V(c_{p_+,p_-})$ is not simple and has a unique non-trivial proper ideal I_{p_+,p_-} generated by a singular vector η of conformal weight $\frac{1}{2}(p_+ - 1)(p_- - 1)$. The quotient vertex algebras $M(p_+, p_-) = V(c_{p+,p_-})/I_{p_+,p_-}$ are called the $N = 1$ minimal model vertex algebras and they are rational, that is, c_2-cofinite and all their modules are semi-simple. The purpose of this talk is to sketch a new proof of the module classification of $M(p_+, p_-)$.

The main idea is to invoke Zhu's algebra. The Zhu algebra $A(V)$ of a vertex algebra V is the associative algebra of zero modes of V acting on vectors (in any module) that are annihilated by modes which lower conformal weight. The simple V-modules are in a 1-1 correspondence with the simple $A(V)$-modules.[1] Additionally the Zhu algebra $A(V)$ of a vertex algebra V can be constructed as a vector space quotient of the vertex algebra.

The image $A(I)$ of an ideal $I \subset V$ is an ideal of the Zhu algebra. In particular, $A(V(c_{p_+,p_-})) \cong \mathbb{C}[T]$ where T is the image of the conformal vector and the ideal is generated by the image of singular vector η. This image must therefore be some polynomial $f(T)$, which yields the presentation $A(M(p_+, p_-)) \cong \mathbb{C}/\langle f(T) \rangle$. The allowed conformal weights of highest weight vectors for the $N = 1$ minimal model vertex algebra are therefore the roots of $f(T)$.

[1] Some mild qualifiers need to be added to make this statement fully rigorous, but they shall not concern us here.

In practice it is very hard to determine the explicit formulae for singular vectors which would be required for computing $f(T)$. This problem can be neatly circumvented by using free field realisations to construct the singular vectors as module homomorphism images of certain highest weight vectors. In particular, these free field realisations yield formulae for singular vectors in terms of the combinatorial data of symmetric functions which can also be used to evaluate the action of the zero modes of singular vectors and thereby determine the roots of $f(T)$.

Anton M. Zeitlin. *Towards the continuous Kazhdan-Lusztig correspondence.*

Recently, continuous series of unitary representations of $U_q(sl(2,\mathbb{R}))$ closed under the tensor product were introduced, thus generating the "continuous" tensor category. These representations, known as "modular double" representations, extensively studied by L. Faddeev, K. Schmüdgen and J. Teschner, are constructed via the unitary representations of the quantum plane and do not have the classical limit. The tensor category of representations of the quantum plane is quite subtle and is related to the properties of the quantum dilogarithm. However, it was shown by I. Ip that one can obtain the tensor category of $ax+b$-group, the affine group of a line (below it is denoted as G) as a classical limit of the tensor category of the quantum plane. Also, the basic structures from the quantum Teichmüller theory can be derived from the tensor products of the representations of the quantum plane, which is very important for the proper understanding of Chern-Simons theory associated with $SL(2,\mathbb{R})$ (see below for more details).

An important problem to think about is the construction of the affine analogues of the aforementioned continuous tensor categories. This can lead to generalization the Kazhdan-Lusztig equivalence of categories to the case of the quantum plane and $U_q(sl(2,\mathbb{R}))$. In other words, we propose that there exists a braided tensor category of representations of the affine algebra $\widehat{sl}_k(2,\mathbb{R})$, equivalent to the braided tensor category of the mentioned above continuous series of $U_q(sl(2,\mathbb{R}))$. One fact supporting this conjecture is that there exists a category of representations of the Virasoro algebra related to the Liouville theory, which is equivalent to the mentioned category of representations of the Virasoro algebra. Therefore, the corresponding category of representations of the affine algebra $\widehat{sl}_k(2,\mathbb{R})$ is a missing piece in this equivalence of categories.

However, a first natural problem to solve is to find unitary representations for the $\widehat{ax+b}_k$, the loop version of the $ax+b$-group with central extension (with the central charge k) for the a-subgroup and compare them and their tensor structure with the representation theory of the quantum plane.

One can construct the unitary representations of $\hat{G}$, which naturally generalize the irreducible representations of G. The corresponding representation space is the L^2 space with respect to the Wiener measure. These representations of $\hat{G}$ are labeled by a certain function. It is proven that certain representations in this class (e.g. when this function is constant) are irreducible.

It is known that the unitary representations of G are closed under the tensor product and there are three types of simple objects in the category of unitary representations of G. It appears that their tensor products decompose as the direct integrals of these "simple" objects. Using the principles discussed in the beginning of this section, one can hope to have the braided tensor category for $\hat{G}$, where the braiding is related to the value of the central charge. I also hope to obtain a differential equation governing the intertwining operators, i.e. the analogue of the Knizhnik-Zamolodchikov equation.

However, the results related to the $\widehat{ax+b}_k$-group are only a part of the main task, namely the construction of the continuous series of unitary representations of $\widehat{sl}_k(2,\mathbb{R})$.

Unfortunately, the standard approach of inducing representation of $\widehat{sl}_k(2,\mathbb{R})$ would not fit the construction, since the resulting modules appear to be nonunitary. Together with Igor Frenkel we used the results obtained for $\hat{G}$ to construct new modules for $\widehat{sl}_k(2,\mathbb{R})$ by means of the Wakimoto-type formalism, using the "currents" corresponding to the Lie algebra elements of $\widehat{ax+b}_k$ and the infinite dimensional Heisenberg algebra free fields. It turned out that the correlators of the resulting $\widehat{sl}_k(2,\mathbb{R})$-currents, defining the pairing in the representation diverge, and therefore we had to describe the scheme of eliminating those divergencies. This led to a very interesting graphical formalism, similar to Feynman diagram technique, where divergences corresponded to 1-loop graphs. The regularization scheme involved dependence on the infinite family of parameters: one parameter for each loop with a given number of vertices. One can generalize this construction to the higher rank case, using the similar procedure, that works for the quantum groups, introduced by I. Frenkel and I. Ip. A necessary question which immediately can be asked is as follows: for which values of regularization parameters the resulting representations will be unitary? Answering this question will involve studying the resulting bilinear form and analyzing it, using the graph formalism, that was a cornerstone of the definition of those representations. However, we do believe that finding the representations forming a braided tensor category will single out the necessary family of unitary representations. Therefore the next question to ask is: what are the intertwining operators for a suggested tensor category? One of the ways of doing that is to construct them in a similar fashion as in the Virasoro case given in the papers of J. Teschner.

Currently, the study of Chern-Simons theories with noncompact gauge groups became very important from the point of view of both mathematics and physics. In particular, it was argued since the late 80s that the canonical quantization of $SL(2,\mathbb{R})$ Chern-Simons theory is connected with both the quantum Teichmüller theory, which is related to the representation theory of the quantum plane, and Liouville theory, related to representation theory of $U_q(sl(2,\mathbb{R}))$. It is known that Liouville theory can be obtained from the $SL(2,\mathbb{R})$ WZW theory by means of the Drinfeld-Sokolov reduction. At the same time, the study of Chern-Simons theory with compact gauge group G showed, that its space of states in the presence of Wilson lines is isomorphic to the space of conformal blocks of the WZW model associated with G. Therefore, we may expect that this rule works in the noncompact case too, and the construction of the continuous series of unitary representations of $\widehat{sl}_k(2,\mathbb{R})$ will provide an important link between the quantum Teichmüller theory, $SL(2,\mathbb{R})$ Chern-Simons, $SL(2,\mathbb{R})$ WZW model and its reduction, the Liouville theory.

Yaping Yang. *Cohomological Hall algebras and affine quantum groups.*

It is a classical theorem of Ringel that the Hall algebra of a quiver Q is isomorphic to the positive half of the quantum enveloping algebra of the Lie algebra associated to Q. In this talk, I will talk about an affine analogue of the above theorem.

The goal of my talk is to give a geometric construction of the affine quantum groups: Yangian, quantum loop algebra and the elliptic quantum groups. For each quiver Q and each cohomology theory A, I will introduce the cohomological Hall algebra (CoHA), as the A-cohomology of the moduli of representations of the preprojective algebra of Q. This generalizes the K-theoretic Hall algebra of commuting varieties defined by Schiffmann-Vasserot. I will describe a family of representations of this CoHA coming from A-homology of Nakajima quiver varieties.

When A is the usual cohomology, the (extended) preprojective CoHA is isomorphic to the Borel subalgebra of Yangian, which is a deformation of the current algebra of the Lie algebra of Q. The preprojective CoHA action on Nakajima quiver varieties is compatible with the actions of Yangian constructed by Nakajima, Varagnolo and Maulik-Okounkov. When A is the K-theory and elliptic cohomology, the (extended) preprojective CoHA is expected to be the Borel subalgebra of quantum affine algebra and elliptic quantum group respectively.

The construction of CoHA works for any cohomology theory. In particular, we obtain new affine quantum groups corresponding to arbitrary cohomology theories.

This talk is based on my joint work with Gufang Zhao.

Alex Weekes. *Highest weights for some algebras constructed from Yangians.*

There is a philosophy, as outlined for example by Braden, Licata, Proudfoot and Webster, that one can do ”Lie theory” for quite general Poisson varieties. Here, the role of the enveloping algebra of a Lie algebra is played by a deformation quantization of the algebra of functions on our variety. There are interesting notions of highest weight theory and of a category O.

In this talk we describe work on such a theory for algebras called truncated shifted Yangians. These algebras conjecturally quantize slices to Schubert varieties in the affine Grassmannian. We outline the construction of these algebras, the combinatorics of their highest weight theory, and some connections to the geometry of Nakajima quiver varieties. This is joint work with Joel Kamnitzer, Peter Tingley, Ben Webster and Oded Yacobi.

Jethro van Ekeren. *Fusion for principal W-algebras.*

I describe joint work with T. Arakawa in which we determine modular properties of characters of certain non-C2 vertex algebras, and apply this to establish the fusion rules of principal affine W-algebras originally computed by Frenkel, Kac and Wakimoto.

Xiao He. *Reduction by stages for affine W algebras.*

We are considering the reduction by stages in the affine W-algebras level. In the finite W-algebra case, as the finite W-algebras are the quantizations of the Slodowy slices and there are reduction by stages for Slodowy slices, so sometimes we can lift this reduction by stages procedure to the algebraic level. The affine W-algebras are quantizations of the Arc spaces of the Slodowy slices and we still have the reduction by stages for these Arc spaces of Slodowy slices, so we believe that in some cases we still can lift this reduction by stages procedure to the algebraic level. This kind of reduction by stages can relates the W-algebras more closely and help us to understand better the structures also the representations of W-algebras.

4 Open Problems Session

Shashank Kanade.

In a joint paper with Matthew C. Russell, we found six new conjectural partition identities of Rogers-Ramanujan-Capparelli type. Of those six conjectures, three are directly related to the representation theory of affine Kac-Moody algebras, and the rest may not have much to do with the representation theory. Below, I'll describe the first three of our conjectures.

Given a non-negative integer n, we say that $n = l_1 + l_2 + \cdots + l_k$ is a partition of n if each l_i is a positive integer with $l_1 \geq l_2 \geq \cdots \geq l_k$. We say that a partition satisfies *Condition* * if $l_i \geq l_{i+2} + 3$ for all i and if $l_i - l_{i+1} \in \{0, 1\}$ implies that $3 | l_i + l_{i+1}$.

Here are the conjectures. For all n the following three identities hold.

1. Partitions of n in which each part is congruent to $\pm 1, \pm 3$ mod 9 are equinumerous with the partitions of n that satisfy Condition *.

2. Partitions of n in which each part is congruent to $\pm 2, \pm 3$ mod 9 are equinumerous with the partitions of n that satisfy Condition * and moreover have their smallest part at least 2.

3. Partitions of n in which each part is congruent to $\pm 3, \pm 4$ mod 9 are equinumerous with the partitions of n that satisfy Condition * and moreover have their smallest part at least 3.

These identities have been verified up to partitions of $n = 1500$.

Consider the affine Lie algebra $g = D_4^{(3)}$. It has three integrable modules of level 3. It is known using the Weyl-Kac character formula and the Lepowsky-Milne numerator formula that the principally specialized characters of these modules are closely related to the q-series one obtains by finding the generating functions of partitions that satisfy the mod 9 congruence conditions. Let $F = \prod_{m \equiv \pm 1 (\mathrm{mod} 6), m \geq 1} (1 - q^m)^{-1}$. Then:

$$\chi(L(\Lambda_0 + \Lambda_1)) = F \cdot \prod_{m \equiv \pm 1, \pm 3 (\mathrm{mod} 9), m \geq 1} (1 - q^m)^{-1}$$
$$\chi(L(3\Lambda_0)) = F \cdot \prod_{m \equiv \pm 2, \pm 3 (\mathrm{mod} 9), m \geq 1} (1 - q^m)^{-1}$$
$$\chi(L(\Lambda_2)) = F \cdot \prod_{m \equiv \pm 3, \pm 4 (\mathrm{mod} 9), m \geq 1} (1 - q^m)^{-1}.$$

The remaining 3 identities don't have symmetric congruence conditions and therefore may not be directly related to the representation theory of affine Kac-Moody algebras.

Further details can be found in paper [KR].

Anton M. Zeitlin.

It is known that the quantum groups for the split reductive real Lie algebras have representations via positive unbounded self-adjoint operators in Hilbert space. These are known as modular double representations (results of L. Faddeev, J. Teschner, I.B. Frenkel and I. Ip). In the case of sl(2,R) it is known that they form a continuous braided tensor category: tensor product of two representations labeled by the continuous parameter decompose as a direct integral of such representations. It is also known that in this particular case there is an equivalent braided tensor category for corresponding Virasoro algebra (arising from the so-called Liouville theory). At the same time, in the paper of I.B. Frenkel and A.M. Zeitlin, the category of representations (continuous series) for affine sl(2,R) is proposed, which is expected to give rise to the braided tensor category for affine sl(2,R), equivalent to the above two. There is also a relation of those tensor categories to SL(2,R) Chern-Simons theory and quantum Teichmueller spaces.

Some open problems are:

1) Establish the correspondence between the proposed three categories of sl(2,R).

2) Construct the tensor product decompositions for modular double representations of quantum groups of higher rank.

3) Construct the equivalent to 2) braided tensor categories for W-algebras and corresponding affine algebras.

4) Find the relation of 2), 3) to Chern-Simons theories for higher rank real reductive Lie groups and higher Teichmueller theory.

5 Moonshine Seminar

After dark, the workshop participants met for an informal "Moonshine Seminar" with the following talks:

Fedor Malikov. *Introduction to chiral de Rham complex.*

Terry Gannon. *Mathieu Moonshine.*

Thomas Creutzig. *Supernatural VOA.*

The seminar focused on the following

Mathieu Moonshine Conjecture. Let V be the sheaf cohomology of the chiral de Rham complex over a K3 surface. Then V is a super vertex operator algebra admitting the action of the Mathieu group M_{24} by automorphisms. The action of M_{24} fixes an $N = 4$ superconformal subalgebra in V.

Elliptic genus as a topological invariant of manifolds was introduced by Ochanine [O] and Witten [W]. Eguchi, Ooguri and Tachikawa conjectured in [EOT] that for K3 surfaces elliptic genus admits the action of Mathieu M_{24} group, fixing the $N = 4$ superconformal subalgebra. Recently Bailin Song [S] showed that H^0 of the chiral de Rham complex over a Kummer surface is the $N = 4$ superconformal vertex algebra. Terry Gannon showed in [G] that twisted elliptic genus of a K3 surface does decompose into an infinite sum of characters of M_{24}.

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