Neostability theory

Bradd Hart (MacMaster University),
Ehud Hrushovski (Hebrew University at Jerusalem),
Alf Onshuus (Universidad de los Andes),
Anand Pillay (University of Leeds),
Thomas Scanlon (University of Berkeley),
Frank Wagner (Université Lyon I)

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1 Overview of the Field

In recent years many unexpected results have been proven using methods from model theoretic stability theory in unstable contexts and this subject has taken center stage in model theory.

Let us start with a brief chronological review of the subject.

Starting in the late sixties and for all of the seventies and eighties, stability theory played a central role in model theory. Inaugurated with Morley's celebrated proof of his theorem on theories categorical in an uncountable cardinality, the theory reached a high degree of sophistication with Shelah's classification theory and was then developed by Shelah, Lascar, Poizat and others into what has arguably been the deepest and most applicable of the branches of model theory.

Stability theory reached an apex with the geometric stability theory of, especially, Hrushovski and Zilber. In the late 1970s Zilber introduced the group configuration in his work on totally categorical theories and then Hrushovski generalized the group configuration theorem well beyond these logically perfect theories. In so doing, it was revealed that structures of algebraic or algebraic geometric origin explain the complexity of some very general theories in which there is no apparent geometry. This approach of analyzing a stable structure according to the geometry of the types was incredibly fruitful with results such as the trichotomy theorem for Zariski geometries by Hrushovski and Zilber and the latter applications to diophantine number theory (including the Mordell-Lang Conjecture for fields of positive characteristic) by Hrushovski and later by Scanlon.

From the late nineties, especially with the work of Kim and Pillay, people have tried to generalize the results from stability theory to a more general class of structures, notably simple theories, those theories for which the basic properties of forking independence from stable theories continue to hold. Although many results have been obtained, it appears that it was the robustness of the types (automorphism invariant classes) that one has in stable theories which allowed for many of the most striking of the previously mentioned results and this robustness is lost when generalizing from stability to simplicity.

Since the early 2000's, developments turned the attention of many model theorists back into stability, or, really, back to the fundamental ideas and objects of study of stability theory, such as definable types. These developments prompted us to organize the first BIRS meeting on the subject in 2009, a meeting which directed many of the multiple approaches into defining certain research fields which are now centers of intense research and productivity, something that was reinforced with this last meeting. We will now mention what has happened with the main subfields.

First, the work Haskell, Hrushovski and Macpherson on *stably dominated types* in algebraically closed valued fields (ACVFs) combined very well with what Shelah's work on *generically stable* types in dependent theories. The combination and shared results of these two areas has developed into a central area of research in model theory with at least three well established approaches. There has been ongoing research on the foundational aspects of generically stable types from the first order point of view (for example ongoing work by Pillay and Tanovic, and Onshuus and Usvyatsov); a very important branch is the theory of invariant and generically stable Keisler measures (Hrushovski, Pillay and Simon, among others); and finally, the topology of generically stable types and applications to geometry (Hurshovski and Loeser). This is an area which promises to have deep consequences both in model theory and other areas of mathematics, which makes the study of the foundations all the more important. For instance, with this last mentioned project of Hrushovski and Loeser on stably dominated types, the theory of stably dominated types has been shown to be perfectly apt for the analysis of p-adic analytic spaces.

Secondly, the theory of non forking in dependent theories has manifested itself in disparate areas of mathematics. For example, the idea that non forking could be developed relative to any good notion of smallness (in the sense of ideals or measure zero sets) together with a generalization of the generically presented group theorem underlies Hrushovski's recent breakthrough in additive combinatorics which is now known as the "non-abelian Freiman theorem". Towsner then used these techniques to give a model theoretic proof of Szemerédi's theorem on the existence of arithmetic progressions in sufficiently dense sets of integers. On the other hand, deep ideas from combinatorics, such as Szemerédi's graph regularity theorem, have been consciously imported into general model theory. We expect this synergy to result in further spectacular results.

The foundational studies of dependent theories has continued, with many ideas and results developing and converging after the 2009 meeting. Shelah now has a "counting of types" characterization of dependent theories, while work towards understanding the different properties of non forking in dependent theories has continued with work by Chernikov, Kaplan, Simon and Usvyatsov. This research have prompted the establishment of NTP$_2$ theories as a very adequate generalization of simple and dependent theories, in the sense that it generalizes both and recent work has shown that in many cases, particularly with the behavior of forking, any result that works in both dependent and simple theories usually works also in NTP$_2$ theories.

Finally, work in stongly dependent theories and theories with finite VC (Vapnik-Chervonenkis) dimension (a notion that has deep implications in the complexity of algorithms and learning techniques) has continued, and recent preprints by Kaplan, Onshuus and Usvyatsov, Kaplan and Simon, and Aschenbrenner, Dolich, Haskell, Macpherson and Starchenko, give evidence that the combination of the notions coming from Shelah's work (dp-rank, burden and weight) and those coming from the study of the VC-rank may combine to produce some interesting results.

2 Recent Developments and Open Problems

The study of dependent theories has continued its fruitfulness, the most recent important development has been the proof of Chernikov and Simon ([**?**]) of uniform definability of types over finite sets in dependent theories. This result implies that types over finite sets have a very stable-like behaviour and, by work of Laskowski, is closely connected to long-standing problems in learning theory . It also prompts a very important question as to whether or not this result is true locally (so whether a family of dependent types over finite sets have uniform definability), which would imply that one can look for dependent-like types in any given theory and have a very important tool to work with. Another important development in dependent theory came from the paper "On non-forking spectra" by Chernikov, Kaplan, and Shelah ([**?**]) where they show enough examples to understand in a much better way the possible characterizations of dependent theories by bounded number of types. They show that a bounded ($2^{2^{\kappa}}$) number of non-forking extensions (for types over a model M of size κ) does not characterize dependent theories, although whether or not the bound $\mathrm{ded}(M)^{\aleph_0}$ works is still open. They also prove that within NTP$_2$ theories the bound $2^{2^{\kappa}}$ does characterize dependence, which also enriches the study of NTP$_2$ theories.

Another important result of NTP$_2$ theories was Chernikov and Kaplan's proof of Kim's Lemma ([**?**]). Not only is this result a very important tool when studying non forking behavior which is now valid in both simple and dependent theories, but it gives very strong evidence towards the idea that things which are true for dependent and simple theories should not only work in NTP_2, but that proving the fact in this context will give what in some sense is 'the right proof'.

On a quite different direction, where instead of generalizing we restrict dependent theories to a very interesting subclass, a lot has happened in the study of dp-rank and VC-density. There have been many different developments in VC-minimal and dp-minimal theories, the applications of the Aschenbrenner, Dolich, Haskell, Macpherson, Starchenko ([**?**] and [**?**]) results keep appearing, and the understanding of dp-rank and theories of finite dp-rank is one of the fastest growing areas in model theory at the time. Some of the more imporant results are the additivity of the dp-rank ([**?**]), all the study of dp-minimal theories by Simon and Goodrick ([**?**] and [**?**]) and the proof of uniform definability of types over finite sets in dp-minimal theories by Guingona ([**?**]) which was later generalized to dependent theories in the paper mentioned above. The main open question in this area is whether or not the dp-rank bounds the VC-density, and whether or not one has a definable version of the Helly number theorem for families of sets defined by dependent formulas.

3 Presentation Highlights

- **Mathias Aschenbrenner** opened the meeting with a talk on VC-density that was both an excellent introduction into the subject, developed initially by Vapnik and Chervonenkis in the context of computational learning theory but with a clear model-theoretic content, and his results with Dolich, Haskell, MacPherson and Starchenko. While the notion of VC dimension is by now well-established in model theory, they also study VC density. The latter not only can serve as explanation of polynomial bounds on the computational complexity of geometric arrangements, but can also be used to define strengthenings of the NIP condition.

- **Pierre Simon** spoke about honest definitions in NIP theories taking up one of the topics from Aschenbrenner's talk and showed that they exist in any NIP theory, relating their existence to earlier results about expanding an NIP structure by a predicate. One of the most remarkable results he discussed was his solution with Chernikov of the conjecture on uniform definability of types over finite sets in NIP theories. As Laskowski had demonstrated earlier, the aptly, though stultifyingly technically, named UDTFS property is closely related to some very difficult and long-standing problems on the existence of compression schemes in learning theory. The Chernikov-Simon theorem represents a major breakthrough.

- Theories without the tree property of the second kind (NTP_2) are a common generalisation of simple and NIP theories, and the more audacious would propose that this is the correct class in which to develop the most general form of stability theory. The talk by **Artem Chernikov** was one step in this direction; he showed that in an NTP_2 theory the ideal of forking formulas is S_1 in the terminology of Hrushovski. This is extremely encouraging, since the recent generalization by Hrushovski of the independence theorem and the stabilizer theorem use the existence of a suitable S_1 ideal.

- **Kobi Peterzil** reminded us that contrary to popular opinion, o-minimal theories need not eliminate imaginaries, if there is no underlying one-dimensional group, and showed that imaginaries of dimension one can be eliminated.

- Indiscernible sets and indiscernible sequences have been a mainstay of stability theory from the very beginning, appearing in essential ways in Morley's proof of his categoricity theorem. Shelah had used indiscernibles with respect to more complicated indexing structures, especially with regard to trees, to study dividing lines in the classification theory of unstable theories. With the lectures of **Joon Kim** and **Lynn Scow** we learned about the connections between generalised indiscernibles and structural Ramsey theory and further applications of these indiscernibles to the problem of distinguishing SOP_2 from SOP_3.

- Dependent (or NIP) theories and simple theories form two of the most important classes of theories generalizing stability. Combinatorially, NIP theories are characterized by the absence of a formula $\phi(x; y)$ and parameters $\{b_\sigma : \sigma \subseteq \mathbb{N}\}$ and $\{a_i : i \in \mathbb{N}\}$ for which $\models \phi(a_i; b_\sigma) \iff i \in \sigma$

which simple theories are those where no formula $\phi(x;y)$ has the tree property: there are parameters $\{b_\rho \ : \ \rho \in \omega^{<\omega}\}$ and $\{a_\sigma \ : \ \sigma \in \omega^\omega\}$ and a natural number k so that $\rho \subseteq \sigma \implies \models \phi(a_\sigma; b_\rho)$ and for any $\rho \in \omega^{<\omega}$ the set $\{\phi(x; b_{\rho\frown\langle j\rangle}) \ : \ j \in \omega\}$ is k-inconsistent. The class of theories with the tree property may be further decomposed into those of the first kind and those of the second kind. As we learned in lectures by **Hans Adler** and **Artem Chernikov**, the class of NTP_2 theories, those without the tree property of the second kind, generalizing simple and dependent theories, enjoy strong stable-like properties.

- In the late 1960s, Keisler proposed measuring the complexity of a theory through the class of ultrafilters with respect to which ultrapowers of the given theory are reasonably saturated. On the face of it, since the definition of this relation is highly set theoretic, one would expect that the resulting partial order to be chaotic. However, the known results, mostly proven by Shelah, suggest that the Keisler order is linear and that breaks occur at meaningful points in the classification theoretic hierarchy (finite cover property, stable, stronger order property-3, *et cetera*). After the burst of initial results and a few scattered theorem proven later (the most recent of which was shown a decade and half ago), only very recently have there been any significant breakthroughs. **Maryanthe Malliaris** reported on several new results including an unconditional identification of a new class in the Keisler order.

- Several speakers addressed problems around the structure of groups whose theories satisfy various model theoretic hypotheses. **Martin Hils** explained his proof, with **Martin Bays** and **Misha Gavrilovich** that in an ω-stable theory with the definable multiplicity property, the property that $\frac{1}{n}p$ has Morley degree 1 for all $n < \omega$ is definable for definable families of Morley degree 1 subsets of divisible abelian groups. This in particular implies that for an irreducible subvariety X of a semiabelian variety over an algebraically closed field, the number of irreducible components of $[n]^{-1}(X)$ is bounded uniformly in n, and moreover that the bound is uniform in families X_t, thus closing a gap in the construction of the bad field by Baudisch, Blossier, Martin Pizarro, and Wagner. **Dugald Macpherson** described his joint work with **Katrin Tent** on the structure of pseudofinite groups with NIP or supersimple theories. **Krzysztof Krupiński** and **Jakub Gismatullin** spoke about their study of definable cocycles as a method to produce interesting examples of groups G for which $G^{\circ\circ}$, the smallest type definable group of bounded index, differs from $G^{\circ\circ\circ}$, the smallest invariant (in a saturated model) group of bounded index. **Itay Kaplan** exhibited the subtlety implicit in attempted generalizations of the Baldwin-Saxl theorem, to wit that in a dependent theory given a uniformly definable family of subgroups of a definable group there is a fixed bound N for which every finite intersection of instances of such subgroups reduces to an intersection of at most N such, to families of type definable subgroups.

- **John Baldwin** spoke about the use of set theory in model theory, paying special attention to questions of absoluteness for key notions in abstract elementary classes (AECs). Many dividing lines in first-order classification theory are defined by set theoretic conditions, for example, stability may be defined by bounds on the cardinalities of type spaces, but they are known to be absolute because they also admit arithmetic characterizations. This tends to fail for other logics. **Monica VanDieren** continued the study of AECs with a lecture on non-splitting. In the first-order context, independence relations were based on non-forking rather than non-splitting, but for AECs forking does not really make sense. This handicap forced VanDieren to develop a better theory of non-splitting which yields new information even for first-order theories.

- From the point of view of first-order logic, a fixed finite model does not admit an interesting model theory since the entire structure may be described by a single sentence. However, if one studies families of finite models as one does with the theories of pseudofinite models or works with a weaker logic, then the methods of infinite model theory apply to the finite. **Cameron Hill** developed the theory of thorn-independence in finite variable logic transposing an independence theory which on the face of it works only for infinite structures to the finite. With his lecture not only did he complete the academic exercise of generalization but he showed that rosiness for finite variable logic admits a computational characterization.

- **John Goodrick** spoke about his joint work with **Byunghan Kim** and **Alexei Kolesnikov** on homology groups of first-order theories. He showed how questions about amalgamations of types are encoded by these groups.

- **Assaf Hasson** reported on his proof with **Alf Onshuus** that Henson's graphs, i.e., the generic countable K_n-free graphs, are symmetrically indivisible: For any finite partition one of the parts contains an isomorphic symmetrically embedded substructure (i.e., any automorphism of the substructure extends to an automorphism of the whole).

- A long-standing conjecture of Kueker asserts that for a countable theory T, if every uncountable model of T is $\aleph_0$-saturated, then T must be categorical in some infinite cardinal. **Predrag Tanović** outlined a proof of an important case of Kueker's conjecture: for dependent theories in which $\mathrm{dcl}(\varnothing)$ is infinite.

- Ultraimaginaries, i.e., classes modulo an invariant equivalence relation, arise in non-simple theories as canonical bases, but also appear in the simple context, for instance as non-orthogonality classes of regular types. However, Ben Yaacov had shown that the usual independence theory cannot be extended to include ultraimaginaries. **Frank Wagner** defined a reasonably behaved subclass, 'tame' ultraimaginaries (which in particular suffice for supersimple theories), proved certain basic properties and a feeble elimination result, and demonstrated how the use of ultraimaginaries can explain certain phenomena in finite rank theories and be used for a generalization to infinite rank.

This workshop was one of the last ones in which not all of the lectures were automatically recorded. Two lectures, those of Simon and Peterzil, were webcast. These recordings have already formed the basis of remote seminars. If some of the other excellent lectures were available, they, too, would have been studied by researchers and students who were unable to attend the meeting.

A particular feature of the conference was the moderated question session on Tuesday afternoon. After initial hesitation, many open questions were posed and discussed, and then assembled in a file on the conference website. The evenings were left free of talks for discussion in small groups, and long breaks between the talks enabled prolonged discussions at the end of each presentation.

4 Scientific Progress Made

Many people have commented to have made concrete progress of their research during the meeting.

In particular, following discussions at the meeting, Krupinski, Tanovic, and Wagner, made progress on a 40 year old conjecture of Podewski, namely that a minimal field (so that any definable set is finite or cofinite) of any characteristic is algebraically closed. Chernikov and Hills were able to prove that certain theories of valued difference fields are NTP$_2$. Ben Yaacov and Chernikiv also reported that after proving an "amalgamation of independent types" theorem for NTP$_2$ theories, they were able to prove that in NTP$_2$ theories equality of Lascar types over an amalgamation base is type-definable (more precisely, if a and b have the same Lascar type then they have Lascar distance at most 3.)

Several other groups of people (Berenstein-Dolich-Vassiliev, Tanovic Gismatulin-Krupinski, Macpherson-Steinhorn), are already writing papers based on results achieved during the meeting, although given the general feedback we have had we expect several more to come after the research started during the meeting matures.

5 Outcome of the Meeting

As expected, this meeting offered a very rich array of results. Three years ago the main achievement of the meeting was making researchers aware of the possibilities in the field. Since much progress has been made in settling the main areas and achieving the first results, all this came together to make this a meeting as fruitful as the first one, but in a more concrete way. For example, the TP$_2$-NTP$_2$ dichotomy which had appeared as a curiosity at the 2009 meeting has by now revealed itself to be a major fault line and classification theory of dependent theories is taking on the robust character of stable classification theory.

This meeting had the effect of solidifying the body of work in neostability theory as a coherent project to discern robust divisions within the class of all theories and to develop stability theoretic methods in an appropriate level of generality.

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