Moduli spaces in conformal field theory and Teichmüller theory

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* Unfortunately Wolfgang Staubach was unable to attend the meeting due to unexpected work commitments.

1 Overview of the Field

The definition of conformal field theory originated around 25 years ago with work of Segal ([11]) and others. Constructing mathematically rigorous examples that satisfy this definition has proved extremely difficult. One successful approach has been to develop and use vertex operator algebra theory. Due to the work of Huang (see for example [2, 3]) and Huang and Kong [4] and others, there are now a wide class of examples in genus 0 and 1. Constructing the higher-genus theory will involve the interplay of algebra, geometry and analysis.

- Algebra: Vertex operator algebras and their representation theory, category theory.
- Geometry: Riemann surfaces and their moduli spaces, infinite-dimensional Teichmüller theory.
- Analysis: Geometric function theory, quasiconformal mappings, infinite-dimensional holomorphy.

Before any construction of higher-genus conformal field theory can be carried out, there must first be a precise and mathematically rigorous definition of conformal field theory. This requires new results on the rigged moduli space of Riemann surfaces with parameterized boundaries and the associated determinant line bundle. Well developed tools from Teichmüller theory and geometric function theory can be brought to bear. Results of Radnell, Schippers and Staubach [5, 6, 7, 8, 9] over the past 9 years have solved a number of these problems. Most recently in [9] we have defined a refined Teichmüller space using a class of refined quasiconformal mappings introduced by Takhtajan and Teo [12] in the plane case. The new Teichmüller space appears to be the natural setting for conformal field theory.

2 Recent Developments and Open Problems

The focus of the team meeting was the following problems which are central to the definition and construction of conformal field theory from vertex operator algebras and furthering the connections with Teichmüller theory.

1. Define the determinant line of a Riemann surface whose boundary is parameterized by a refined quasisymmetry.
2. Construct local trivializations of the determinant line bundle over the rigged moduli space using "Faber polynomials", a canonical basis for the set of holomorphic functions on a domain.
3. Construct projectively flat connections on the determinant line bundle.
4. Give a rigorous and complete definition of conformal field theory and a modular functor.
5. Establish a conjectured "sewing property" for suitable "meromorphic functions" on the rigged moduli space.
6. Using the "sewing property" for these "meromorphic functions" on the rigged moduli space to show that traces of products or iterates of intertwining operators for suitable vertex operator algebras satisfy differential equations with regular singular points and thus are absolutely convergent.

3 Presentation Highlights

Being a "research in teams" meeting we did not give any formal presentations. However, because each of the participants works in a different area of mathematics, informal presentations were given at the beginning of the workshop to quickly bring each other up to date on the most recent developments and open problems. These were: (1) Schippers, "Faber Polynomials", (2) Huang, "meromorphic function on rigged moduli space", and (3) Radnell, "The definition of conformal field theory".

4 Scientific Progress Made

The main aim of the meeting was to bring together the recent work of the participants and use this to precisely formulate problems and the method of solution. We were very successful in this regard. These long standing problems have now been made very explicit and we have immediate approaches to solving them.

Refer to the list above for the statement of the problems.

1. Analysis problems involving curve regularity and the solution to the jump problem have recently been solved by two of us (Schippers and Staubach). The approach used by Huang in [1] to construct the determinant line can now be emulated in our setting of refined quasisymmetries. In genus-zero this problem has thus essentially been completed.
2. In the genus-zero case the classical Faber polynomials give a canonical basis for the space of holomorphic functions on a given domain. Three of us (Radnell, Schippers, Staubach [10]) shortly after the meeting used these together with the classical Grunsky matrices to construct global trivializations of the bundle of holomorphic maps over the rigged moduli space. This results in an alternate description of the determinant line bundle in genus zero, which can be generalized to higher genus (see next item).

   For higher-genus Riemann surfaces, analogues of these Faber polynomials exist in the literature (Tietz, [14]). During the meeting we completed most of the details involved in using these results in our setting. This has given us an explicit description of cokernel of the operator used to define the determinant line bundle, which was previously not possible to obtain. The determinant line bundle for higher-genus Riemann surfaces can therefore also constructed in a direct way.
3. We discussed the precise meaning of the conjecture/folk theorem on the existence of a projectively flat connections on the determinant line bundle and modular functor. Now that we have a precise formulation of the underlying spaces, there are many tools from Teichmüller theory available to tackle this problem.
4. The geometric category underlying the definition of conformal field theory is the moduli space of the Riemann surfaces with parameterized boundaries. Using the refined quasisymmetric mappings of Takhtajan and Teo [12] and our recently defined refined Teichmüller space [9] we now have the precise

moduli space on which the rest of the definition of conformal field theory can be built. We now have a complete and rigorous definition of conformal theory and also a holomorphic modular functor.

5. During the meeting we discovered that the Faber polynomials, and their higher-genus analogues, have precisely the properties needed to define the class of "meromorphic functions" of interest. At first it seems coincidental that Faber polynomials appear in this seemly unrelated problem. However, the underlying reason is that the definition of the Faber polynomials and their generalization use the data of a boundary parametrization of the Riemann surface. Formulating things this ways enabled us to interpret these function not just as function on the Riemann surface but actually as functions on the rigged moduli space. The sewing property for these "meromorphic functions" remains to be proved. We discussed a number of approaches which will be explored in the immediate future.

6. Showing the convergence of the traces requires the solution of the previous problem. Long before the meeting, Huang obtained a clear approach based on the genus-one result of Huang [2, 3]. In order to apply this method to higher-genus, the solution of the previous problem is required. Our newly discovered approach in the above item may make it possible to prove the convergence of the traces.

5 Outcome of the Meeting

Due to the work of the team members over the past years we are now at the stage to be able to complete the main problems as listed above. Due to the amount of work to be done this will take some time. At the meeting, we solved problems 1, 2 and 5, and have some new insight as to how to proceed on problems 3, 4 and 6. So far, the meeting has resulted in one publication [10] to be submitted shortly and several others are in preparation. We have planned to meet again in summer 2013.

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