Character Varieties in the Geometry and Topology of Low-dimensional Manifolds (10w5094)

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1 Overview and Introduction

The workshop (the first of its kind) brought together researchers who use character variety methods and related techniques to study the geometry and topology of low-dimensional manifolds.

We briefly recall some background. Suppose that H is a finitely generated group and G a real or complex algebraic Lie group. The set of representations of H in G, denoted $R_G(H)$, can be given the structure of a real or complex algebraic set called the *G-representation variety of H*. In addition, the algebro-geometric quotient of $R_G(H)$ by G determines an algebraic set $X_G(H)$ called the *G-character variety of H*.

The main areas that were focused on were:

- Teichmüller theory and connections to the $SL(2, \mathbf{C})$- character variety of a surface group;
- surface group representations with more general Lie groups G as targets;
- applications of the $SL(2, \mathbf{C})$-character variety to the study of bounded 3-manifolds;
- connections between $SU(2)$ character varieties, gauge theory, and 3-manifold topology;
- the role of the character varieties in deformations of geometric structures.

Rather than having a specific set of open problems to focus on, one of the motivating goals of the workshop was to foster a better understanding of the different aspects of the character variety as well exposing both junior and senior people to the panorama of mathematics that exists in the study of character varieties in connection with the topics mentioned above.

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2 Recent Developments

Let G be $SL(2,\mathbf{R})$ or $SL(2,\mathbf{C})$ and let H be the fundamental group of a compact, orientable, hyperbolizable 3-manifold M with incompressible boundary containing a closed orientable surface Σ of genus at least 2. When $M=\Sigma\times I$, H is a surface group and $X_G(H)$ has been of central importance in the study of the geometry of M and Σ over many decades. Discrete embeddings of H in G correspond to hyperbolic structures and display remarkable flexibility. These ideas, which form a part of Teichmuller theory, were subsequently vital to Thurston's work on geometrization of 3-manifolds. The last few years have seen spectacular developments in our understanding of the topology of the general M, such as the solutions of the Tameness Conjecture (by Agol, and independently Calegari-Gabai) and of the Ending Lamination Conjecture (by Brock-Canary-Minsky). This work leads to a classification of hyperbolic manifolds homotopy equivalent to M which, in turn, gives a better understanding of part of the $SL(2,\mathbf{C})$-character varieties of these groups H. Other recent developments regarding the $SL(2,\mathbf{C})$-character variety can be found in the proofs of the orbifold theorem by Boileau-Leeb-Porti and Cooper-Hodgson-Kerckhoff. We were fortunate in that Brock, Canary, Hodgson, Kerckhoff Minsky and Porti all attended the workshop (Boileau unfortunately had to cancel at the last minute because of disruption to air travel caused by a volcanic eruption).

For groups G more complicated than $SL(2,\mathbf{C})$, the G-character variety of Σ also has special properties, such as components corresponding to the Hitchin-Labourie-Fock-Goncharov "higher Teichmuller spaces". These representations correspond to more general geometric structures, such as real projective structures. There has been a lot of recent work in this area, and one of the leaders in the field (A. Wienhard) attended the meeting.

Next let $G=SL(2,\mathbf{C})$ and suppose H is the fundamental group of a non-compact, orientable, finite volume hyperbolic 3-manifold. Embedded, incompressible, non-boundary-parallel surfaces in compact 3-manifolds are one of the most important structural tools in 3-manifold topology. However, producing such surfaces remains an elusive problem. The seminal work of Culler and Shalen on the $SL(2,\mathbf{C})$-character variety of such groups yields a general method of producing them, and in particular, ones with non-empty boundary. Their work remains one of the few general methods for constructing such surfaces. This subsequently led to the notion of the A-polynomial, a powerful invariant of a knot.

Culler and Shalen as well as several other of the main people who were at the forefront of these developments attended.

3 Presentation Highlights

The organizers selected five participants to each give 2 lectures. These reflected the main themes of the workshop and had both an introductory part as well as a discussion of more recent developments. These were:

Hans Boden: *Connections between $SU(2)$ character varieties, gauge theory, and 3-manifold topology.*

Ken Bromberg: *Teichmuller theory and the character variety.*

Daryl Cooper: *The role of the character variety in deformations of geometric structures.*

Peter Shalen: *The character variety and the topology of 3-dimensional manifolds.*

Anna Wienhard: *Surface group representations in Lie groups.*

As well as these lectures, there were lectures by both junior and senior people that illustrated recent developments in the various fields. The senior people giving talks were:

Tsachik Gelander (*On the dynamics of $Aut(F_n)$ on character varieties*), Ursula Hamenstadt (*Bounded cohomology: Its construction and use*), Yair Minsky (*Primitive stability and the action of $Out(F_n)$ on character varieties*) and Joan Porti (*Regenerating hyperbolic cone 3-manifolds from dimension 2*).

Junior people giving talks were:

Shinpei Baba (2π-*graftings on complex projective structures*), Melissa Macasieb (*On character varieties of 2-bridge knot groups*), Aaron Magid (*Local connectivity of deformation spaces of Kleinian groups*), Stephan Tillmann (*Representations of closed 3-manifold groups*)

Some particular highlights of the lectures were Wienhard's lectures that described recent work on giving a geometric interpretation to representations in the Hitchin component for $SL(4, \mathbf{R})$ that generalizes work of Goldman in the case of $SL(3, \mathbf{R})$.

Other highlights include the lectures of Gelander and Minsky that discussed the dynamics of the action of $Aut(F_n)$ and $Out(F_n)$ on $X(F_n)$. In particular, Minsky described his work on the existence of an open subset of $X(F_n)$ that properly contains the so-called Schottky characters which is $Out(F_n)$ invariant, and on which $Out(F_n)$ acts properly discontinuously.

4 Scientific Progress Made

The meeting proved to be a fertile ground for people from a variety of backgrounds who study the character variety. Many of the attendee's are now involved in some way, as part of a proposed Research Network in Topology, Geometry, and Dynamics of Character Varieties submitted to NSF.