Random Matrices, Inverse Spectral Methods and Asymptotics

Estelle Basor (American Institute of Mathematics),
Marco Bertola (Concordia University),
Betrand Eynard (SPhT CEA Saclay),
John Harnad (Concordia University and Centre de Recherche Mathematique),
Alexander Its (Indiana University - Purdue University at Indianapolis),
Ken McLaughlin (University of Arizona)

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Participants at the workshop ranged over a number of different fields, ranging from theoretical physics and random matrix theory through algebro-geometry and integrable systems, to asymptotic analysis in the complex domain. We will start with an overview of a few of the fields represented at the workshop.

1 Overview of the Field

The workshop's two main highlights were Random Matrices and Spectral Methods: in fact the methods that are employed in the treatment of both subjects have many areas of overlap. The mathematics that has been developed for Random Matrix Theory in the past two decades is astonishingly rich and includes variational techniques, inverse spectral methods as applied to nonlinear integrable differential and difference systems, new asymptotic techniques, such as the nonlinear steepest descent method, free probability and large deviations methods. The results obtained have found new applications in a stunningly wide range of areas of both mathematics and theoretical physics such as, for example, approximation theory, orthogonal polynomials and their asymptotics, number theory, combinatorics, dynamical systems of integrable type, representation theory of finite and infinite groups, growth phenomena, quantum gravity, conformal field theory, supersymmetric Yang-Mills theory and string theory. The principal goal of Random Matrix Theory (RMT) is the description of the statistical properties of the eigenvalues or singular values of ensembles of matrices with random entries subject to some chosen distribution, in particular when the size of the matrix becomes very large. The prototypical example consists in the study of the ensemble of $N \times N$ Hermitean matrices M with a probability measure invariant under conjugation by unitary matrices. The simplest such class (and the most studied) consists of statistical models where the probability measure can be written as

$$\mathrm{d}\mu(M) := \frac{1}{Z_N} \mathrm{e}^{-\Lambda \mathrm{Tr} V(M)} \mathrm{d}M \tag{1}$$

with $V(x)$ –a scalar function– called the **potential** and Λ a (large) parameter. The normalizing factor $Z_N = Z_N[V, \Lambda]$ is called the *partition function* and plays crucial role in the combinatorial part of the theory for its connections to enumerations of ribbon graphs (see the talks of Pierce and Ercolani).

When studying the statistics of the eigenvalues, Mehta and Gaudin [8, 7] showed that all the information can be extracted from the knowledge of the associated orthogonal polynomials

$$\int_{\mathbb{R}} p_n(x) p_m(x) \mathrm{e}^{-\Lambda V(x)} \mathrm{d}x = h_n \delta_{nm} \ , \ \ h_n > 0, \ \ p_n(x) = x^n + \dots \tag{2}$$

and the associated **kernel**

$$K_N(x,x') := \mathrm{e}^{-\frac{\Lambda}{2}(V(x)+V(x'))} \frac{1}{h_{N-1}} \frac{p_N(x)p_{N-1}(x') - p_{N-1}(x)p_N(x')}{x-x'} \tag{3}$$

out of which all correlation functions $\mathbb{P}_N(\lambda_1,\dots,\lambda_k)$ for k eigenvalues from the spectrum of M can be computed in terms of determinants [7].

$$\mathbb{P}_N(\lambda_1,\dots,\lambda_k) = \det\left(K_N(\lambda_i,\lambda_j)\right)_{1\le i,j\le k}, \tag{4}$$

The first breaktrhough came when it was recognized [6] that the above orthgonal polynomials can be characterized in terms of a **Riemann–Hilbert problem**; the second one came with the application of the **nonlinear steepest descent** method [1, 3, 4] to this Riemann–Hilbert problem, since it led to results about the universal properties of the kernel in various scaling regimes.

The statistical distributions that occur in this regime of large sizes display some features which are very ”robust” in the sense that they appear rather independently of the distribution chosen for the matrix entries. This phenomenon goes under the general heading of “universality” and it is not conceptually dissimilar from the more commonly known central limit theorem. An example of these results is that the largest eigenvalue distribution has been shown to possess a limiting distribution (the Tracy-Widom distribution), expressible in closed form in terms of the Hastings-McLeod solution of the Painlevé II equation [10]. This was first established in the case of the Gaussian unitary ensemble of random matrices (i.e. $V(x) = x^2$), but was later extended to even quartic potentials [1] and real analytic potentials [2], using [3]. The form of the asymptotic result is:

$$\lim_{N\to\infty} \mathbf{Prob}\left(\lambda_{\max} < \beta + cN^{-2/3}s\right) = F_{TW}(s) \tag{5}$$

where the constant c depends on the external field V, and $F_{TW}(s)$ is the famous Tracy-Widom distribution, independent of the specific form of V.

1.1 Inverse spectral theory

RMT can be thought of as an application of the study of spectra of large operators in the “forward” direction; its converse application is what underlyies the area of “inverse spectral methods”.

The simplest example is the Korteweg–de Vries equation (and its associated hierarchy), determining the evolution in ”time” of the potential $u(x,t)$ in the Schrödinger equation

$$\mathcal{L} := -\partial_x^2 + u(x,t) \tag{6}$$

in such a way that the spectrum of $\mathcal{L}$ as an operator on $L^2(\mathbb{R}, \mathrm{d}x)$ is independent of time. The evolution of $u(x,t)$ is nonlinear according to the celebrated KdV equation

$$u_t + u_{xxx} + u\,u_x = 0 \tag{7}$$

which describes propagation of (nonlinear) waves in shallow water (in a uni-dimensional approximation); the support of the spectrum of the associated *Lax operator* $\mathcal{L}$ is preserved, but some data in the so–called *scattering data* evolve according to a linear equation and in a simple way.

Therefore, while the original evolution of u is nonlinear, the *scattering data evolution* is “trivial” and all the nonlinearity is hidden in the map that associates to each potential $u(x,t)$ (where t figures as a parameter) the spectral data, and viceversa.

In particular, the “viceversa” direction, namely the **inverse spectral transform** can be achieved in terms of the solution of an integral equation (Gel’fand-Levitan-Marchenko), which can also be recast into an appropriate 2×2 Riemann–Hilbert problem.

We see here where the point of contacts of RMT and ISM lie; the techniques deployed to analyze the relevant Riemann–Hilbert problems in asymptotic regimes are identical on a “philosophical” level, with details of implementation that are understandably of a quite different nature.

The evolution of the KdV equation is just the first example of many other nonlinear evolution equations. The second fundamental example is the semi-classical analysis of the focusing nonlinear Schrödinger equation:

$$i\epsilon\psi_t + \frac{\epsilon^2}{2}\psi_{xx} + |\psi|^2\psi = 0, \qquad \psi(x,0) = A_0(x)e^{iS_0(x)/\epsilon}. \tag{8}$$

The formal WKB guess, $\psi(x,t) \sim A(x,t)e^{iS(x,t)/\epsilon}$, leads to a coupled system of pdes for A and S that are *elliptic* rather than hyperbolic. Thus, although the NLS initial value problem is well-posed in standard Sobolev spaces, the singular limit $\epsilon \to 0$, raises the fundamental analytical question: how does the NLS initial value problem regularize an ill-posed singular limit?

1.1.1 The Connection to Orthogonal Polynomials and Hankel determinants

The partition function of matrix models

$$Z_N[V] := \int \mathrm{d}M \mathrm{e}^{-\Lambda \mathrm{Tr} V(M)} \tag{9}$$

is intimately connected to the theory of orthogonal polynomials, as we briefly recall below. Here, $\mathrm{d}M$ stands for the standard Lebesgue measure on the vector space of Hermitean matrices of size $N \times N$. The parameter Λ that appears in these formulas is a convenience *scaling parameter*: in the study of the model for large sizes N of the matrices, one concurrently sends Λ to infinity in such a way that N/Λ remains bounded. For simplicity we take V -the *potential*- to be a polynomial

$$V(x) = \sum_{j=1}^{\nu} \frac{t_j}{j} x^j \ . \tag{10}$$

Consider the measure

$$w_\Lambda(x)dx := \exp\left[-\Lambda V(x)\right]dx. \tag{11}$$

Let us define $\{p_j(x;\ N,\ \mathbf{t})\}_{j=0}^{\infty}$ to be the sequence of polynomials orthogonal with respect to the measure $w_\Lambda(x)dx$. That is, $\{p_j(x;\ N,\ \mathbf{t})\}_{j=0}^{\infty}$ satisfies

$$\int_{-\infty}^{\infty} p_j p_k w_N dx = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases} , \tag{12}$$

and $p_j(x;\ N,\ \mathbf{t}) = \gamma_j^{(N)} x^j + \cdots$, $\gamma_k^{(N)} > 0$. (The leading coefficient $\gamma_k^{(n)}$ is of course dependent on the parameters $t_1, \ldots, t_\nu$, however we suppress this dependence for notational convenience.) The fact of the matter is that $Z_N(\mathbf{t})$ may also be defined via

$$Z_N(\mathbf{t}) = N! \prod_{\ell=0}^{N-1} \left(\gamma_\ell^{(N)}\right)^{-2} . \tag{13}$$

Z_N is also defined via

$$Z_N(t_1, \ldots, t_\nu) = N! \begin{vmatrix} c_0 & c_1 & \cdots & c_{N-1} \\ c_1 & c_2 & \cdots & c_N \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ c_{N-1} & c_N & \cdots & c_{2N-2} \end{vmatrix} , \tag{14}$$

where $c_j = \int_{\mathbb{R}} x^j w_N(x)dx$ are the moments of the measure $w_N(x)dx$, and the determinant above is called a Hankel determinant (see, for example, Szegö's classic text [9]). The asymptotic expansion (17) constitutes a version of the strong Szegö limit theorem for Hankel determinants. The strong Szegö limit theorem concerns the asymptotic behavior of Toeplitz determinants associated to a given measure on the interval $(0, 2\pi)$ (see [9] for more information).

Amongst the main achievements of the past decade or so are;

• the development and deployment of the nonlinear steepest descent method and Riemann–Hilbert method for the study of asymptotic properties in RMT and orthogonal polynomials;

• The discovery that the Tracy-Widom probability distributions originally arising in random matrix theory are also present in the asymptotic statistical behavior of longest increasing subsequence problems in combinatorics, as well as in the limiting statistics of random tiling problems, random growth processes, interacting particle systems and queueing theory;

• the discovery and proof of the universality classes of the sine, Airy and Bessel kernels;

• the connection between matrix integrals and the enumeration of graphs on surfaces;

• the relation of partition functions and spacing distributions to tau functions in integrable systems and isomonodromic deformations;

• the surprising coincidence of the distributions in the Gaussian Unitary Ensemble and the nontrivial zeroes of the Riemann zeta function;

• the spectral duality in multi–matrix models;

• the connection between large N limits, dispersionless hierarchies, critical limits and minimal CFT (Conformal Field Theory).

1.2 Combinatorics, Quantum gravity, Liouville CFT

In the context of combinatorics, Matrix integrals were introduced in 1974 (t'Hooft) and in the 80's (in the work of Brezin Itzykson Parisi Zuber, David, Ambjorn, Kazakov, for instance (note this is by no means a complete list)) as a tool for counting discrete surfaces, and led to a domain of physics called 2D quantum gravity, or CFT coupled to Liouville gravity, which is, at its core, the combinatorics of maps of given genus. CFT and Liouville theory are still very active topics in physics, and many new results have been obtained recentely about boundary operators (Zamolochikov). The fact is (following t'Hooft and later Brezin Itzykson Parisi Zuber) that the formal large N expansion of matrix integrals is the generating function for the enumeration of maps of a given genus. Thus, 2D gravity and CFT consist in computing the large N expansion of formal matrix integrals, and several progress have been made. It was understood in 1995 (Ambjorn, Chekhov, Kristjansen, Makeenko) how to extract in principle the large N expansion from the loop equation method, and the link with algebraic geometry was progressively uncovered (Kazakov Marshakov, Chekhov, Mironov, Dijkgraaf, Eynard, Bertola, Kostov, and so many others), and some recent progress 2004 (Eynard, Chekhov, Orantin), where some simple explicit formulae for this expansion in terms of algebraic geometric symplectic invariants of the spectral curve were found. On the more mathematical side of things, in 2003 Ercolani and McLaughlin proved that in some region of the parameter space, the large N expansion of the formal matrix integral is convergent and coincides with the actual (not formal) matrix integral.

2 Recent Developments and Open Problems

Recent developments include

• The discovery that random tri-diagonal matrices can produce any member of the so-called general β ensembles.

• The proof of existence of a limiting density of states near the "Wigner semicircle" for banded random matrices using super-symmetric techniques by Disertori, Pinson, and Spencer.

• The application of techniques, originally developed for the analysis of integrable systems, to orthogonal polynomials, approximation theory, and random matrix theory.

• The discovery that probability distributions originally arising in random matrix theory are also present in the asymptotic statistical behavior of longest increasing subsequence problems in combinatorics, as well as in the limiting statistics of random tiling problems, random growth processes, interacting particle systems and queueing theory.

• The emergence of these distributions as universal distribution functions, appearing repeatedly in unexpected areas, ranging from spacing between cars when parallel parked, to waiting times in actively controlled transportation (bus) systems.

• The insight that determinantal particle processes form a strong link between random matrices and representation theory

• The recent flood of conjectures involving the Riemann zeta function and integrals arising in random matrix theory.

• The establishment of universality for local eigenvalue spacings for symmetry classes other than GUE.

Open areas of research include:

• analysis of nonintersecting brownian motion;

• higher order critical phenomena in random matrices;

• statistics of eigenvalues of sample covariance matrices with non-null covariance matrix random matrices with external source (beyond the Gaussian case);

• coupled random matrix theory;

• asymptotics of multiple orthogonal polynomials

• random tiling problems with boundary

• asymptotics for β ensembles

Very recent and at times tenuous connections can be found in the following areas:

• mathematical physics and the Schramm-Loewner Evolution equations

• statistical physics and the Razumov–Stroganov conjecture

• topological string theory, random matrices and QCD

3 Presentation Highlights

The fifteen hour-long contributions can be grouped in the following areas:

1. Riemann–Hilbert methods applied to nonlinear partial/ordinary differential equations (Buckingham, DiFranco, Jenkins, Miller, Niles);
2. Orthogonal polynomials in the plane (Balogh, Putinar);
3. applications to graphical enumerations (Ercolani, Pierce, Prats Ferrer);
4. multiple orthogonality and applicaton to multi-matrix models (Gekhtman, Lee, Szmigielski);
5. fermionic methods for computation of multiple integrals (Harnad, Wang);
6. Probabilistic methods on non-invariant ensemble (Soshnikov).

Due to Visa problems we regret the absence of Irina Nenciu, who had to forfeit at the last minute.

(1) Riemann–Hilbert methods

The talks were divided into applications of RH methods to classical ODEs of Painlevé type (Buckingham, diFranco, Niles) and the dispersionless asymptotics for the Nonlinear Schroedinger equation (Jenkins and Miller).

The Painlevé functions, or transcendents, are solutions of certain nonlinear differential equations; in applications to random matrix theory and two–dimensional statistical models it is sometimes necessary to compute integrals of the transcendent. For example, in computing the asymptotics of the Tracy–Widom distribution mentioned above (5) the behavior has been obtained in the original work, up to an overall constant, whose value was conjectured but not proved. Such constant is precisely related to one such integral of a Painlevé transcendent and its determination is an important technical achievement. In other instances of such integrals, their value is not known, not even conjecturally. The talks by Buckingham and diFranco brought to the attention of the audience some new methods to obtain these values by clever manipulations of the associated linear problem, thus providing an important general mindset in approaching such problems.

The methods rely upon the nonlinear steepest descent approach, the same that can be used in the study of the dispersionless asymptotics of the nonlinear Schroedinger equation; here, Miller's talk showed some applications to experimental physics and what a theoretical approach can predict.

(2) Orthogonal polynomials in the plane These are holomorphic polynomials which are orthogonal with respect to a measure on the complex plane typically supported on subsets of positive Lebesgue measure. There are two main lines of research. On one side are Bergmann polynomials, namely polynomials which are orthogonal w.r.t the Lebesgue measure restricted to some (possibly multiply connected) domain $\mathcal{D}$;

$$\int_{\mathcal{D}} p_n(z)\overline{p_m(z)}\mathrm{d}^2 z = \delta_{nm} \tag{15}$$

Important questions are related to the characterization of the asymptotic distribution of the roots of these polynomials and how this may depend on the geometry of $\mathcal{D}$ and how to effectively reconstruct the shape of $\mathcal{D}$ from the knowledge of a finite number of the moments of the measure. An **archipelago** is the term used by Putinar and collaborators to denote (in a picturesque form) a domain consisting of several connected components. A typical example is the domain consisting of several disjoint disks. For such an archipelago, the talk of Putinar showed that –depending on the geometry– the zeroes distribute along one–dimensional arcs, which may be outside of $\mathcal{D}$ (but always in its convex hull) and how the shape of $\mathcal{D}$ can be reconstructed (in approximate form) from the knowledge of the moments or –which is equivalent– the orthogonal polynomials.

Of a similar tone was Balogh's talk, which reported on recent progress in establishing the validity of certain outstanding conjectures regarding the distribution of zeroes of OPs and relationships with the harmonic measure of domains. Here the setting is different inasmuch as the polynomials are orthogonal w.r.t. to a *weight* on $\mathbb{C}$, namely a positive function with sufficient decay at infinity. In contrast to the above setting of the archipelago, there is no intrinsic geometrical input from the start. However, for weights of the form

$$w(z,\overline{z}) = \mathrm{e}^{-\Lambda(|z|^2 + h(z,\overline{z}))}\ , \quad h(z,\overline{z}) = \text{ harmonic}$$

it is conjectured that the roots of the polynomials as $n \to \infty, \Lambda \to \infty, n/\Lambda = \mathcal{O}(1)$ distribute on arcs that constitute the so–called *mother body* of a domain related to the choice of function $h(z,\overline{z})$. Balogh's talk reported on recent progress made in verifying such conjecture for the particular choice of $h(z,\overline{z}) = \beta \ln|z-a|$, where the asymptotic domain is related to the Youkowsky airfoil. This is an important result because it proves to be the first rigorous such proof of the conjecture, using Riemann–Hilbert methods.

(3) Application to graphical enumerations

The connection between the partition function of matrix models and enumeration of graphs on Riemann–surfaces was in fact one of the first breaktrhough that showed the breadth of applicability of matrix models [5].

For example, a partition function of the form

$$Z_N = \int \mathrm{d}M \mathrm{e}^{-N\mathrm{Tr}(M^2 + t_4 M^4)} \tag{16}$$

admits an asymptotic expansion

$$\log\left(\hat{Z}_N\right) = N^2 e_0(t_4) + e_1(t_4) + \frac{1}{N^2} e_2(t_4) + \cdots\ , \tag{17}$$

where $\hat{Z}_N = Z_N / Z_N(0)$. The coefficients in the Taylor expansions of the functions $e_g(t_4)$ count the number of graphs with only four–valent vertices that can be drawn on a Riemann surface of genus g. This can be generalized to arbitrary valence by adding corresponding terms $t_j M^j$ in the exponen of (16).

The expression is also a solution of the Toda hierarchy, namely of certain nonlinead PDEs: the information from this latter can be used to deduce combinatorical identities for the coefficients of the expansion (17) and Ercolani's talk reported on recent progress in this direction.

In Pierce's talk a generalization was presented whereby the matrix integral in (16) is replaced by integration over real–symmetric or symplectic matrices. The resulting partition functions solve a different hierachy of PDEs that goes under the name of "Pfaffian lattice equations". A universality between all three ensembles of random matrices can be established; as a consequence the leading orders of the free energy for large matrices agree (up to a rescaling of the parameters). Also, Pierce showed an explicit formula for the two point function F_{nm} which represents the number of connected ribbon graphs with two vertices of degrees n and m on a sphere, basing the derivation on the Faber polynomials (and its Grunsky coefficients) defined on the spectral curve of the dispersionless Toda lattice hierarchy.

With different methods and scope, Prats Ferrer's talk showed the method of the "loop equations" in the computation of the above-mentioned combinatorial generating functions in very general models involving not one, but several matrices coupled in chains, and the limit where this chain becomes a continuum.

(4) Multiple orthogonality and application to multi–matrix models

Multi–matrix models are a (relatively) new frontier of random matrices; applications are to refined enumerations of *colored* ribbon graphs and new universality results. A new model where the interaction between the matrices is of determinantal form has been introduced by Bertola, Gekthman and Szmigielski. Szmigielski showed the origin of the model in the connection with the inverse spectral problem for the "cubic string", namely the boundary value problem for the ODE

$$\Phi'''(\xi) = -zm(\xi)\Phi(\xi) \tag{18}$$

where z is the eigenvalue parameter and the problem is that of reconstructing $m(\xi)$ from the knowledge of the eigenvalues (with suitable boudary values) of the above equation. This appeared in the study of deGasperis–Procesi *peakons*, namely nonsmooth soliton solutions of the homonymous nonlinear wave equation. Gekhtman showed how the problem is connected to matrix models and a new class of biorthogonal polynomials, satisfying

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} p_n(x)q_m(y)\frac{\alpha(x)\mathrm{d}x\beta(y)\mathrm{d}y}{x+y} = \delta_{nm} \tag{19}$$

for arbitrary positive measures $\alpha(x)\mathrm{d}x, \beta(y)\mathrm{d}y$ on the positive axis. Their main properties are very close to corresponding ones in the classical theory of orthogonal polynomials; simplicity of the roots, interlacing properties and total positivity of the moment matrix. A characterization of Cauchy BOPs in terms of a 3 by 3 matrix Riemann- Hilbert problem was detailed.

The Dyson model describes the random walk of particles on the line subjected to mutual repulsion; consequently the walks are self-avoiding. Lee's talk recalled how the model can be phrased as a two–matrix model with exponential interaction and how the gap probabilities can be interpreted in terms of isomonodromic theory à la Jimbo-Miwa-Ueno. Once more, the connection is established through the reduction of the model to suitable biorthogonal polynomials. This important contribution shows that the gap probability solve a nonlinear PDE with the Painlevé property.

(5) Fermionic methods The talks of Harnad and Wang showed the use of Fermionic methods in the investigation of matrix integrals. One considers a matrix integral of the type illustrated above (and more general ones) as *expectation values* of (possibly formal) operators on the Fermi–Fock space, namely the exterior space modeled on some abstract (but sometimes quite concrete) separable Hilbert space. Similar methods are used by Okounkov in dealing with the statistical properties of "melting chrystals" or random stepped surfaces. Harnad's talk showed the wide encompassing results that can be demonstrated by an appropriate choice of the operator whose expectation value is being computed; depending on the choice of operator, a very large class of matrix models and multi-matrix model can be recast in this framework. This yields, as an immediate consequence, a method to show that partition functions are Kadomtsev–Petviashvili tau functions (or generalizations thereof) and hence one may derive hierarchies of partial differential equations. In this vein, Wang's talk illustrated this very general statement in the particular case of the Wishart ensemble, which was not immediately captured in the general setting of Harnad's talk.

(6) Probabilistic methods Soshnikov was the sole representative of probability theory and its applications to random matrices; his talk discussed the Wigner ensemble. This is a random matrix ensemble which is not invariant under the adjoint action of the unitary group. Wigner matrices are random matrices whose entries are independently (and possibly identically) distributed; only in the case of iid *normal* (i.e. Gaussian) variables the ensemble has the unitary symmetry, and in this case it is amenable to the "usual" Hermitean model as discussed above. As soon as the distribution is not normal, new challenges arise since it is not possible to write a closed form for the induced probability distribution on the spectrum. New methods of free probability and large deviation techniques must be deployed.

4 Scientific outcome

The group of participant was very focused and homogeneous in interests, which facilitated the cross-interaction between the participants. True to the empirical theorem of the six degrees of separation, all participants were separated at most by three steps in collaborative distance and had known each other from previously. Several had ongoing collaborations (Bertola-Gekhtman-Szmigielsky, Buckingham-McLaughlin-Miller, diFranco-Miller, Ercolani-Pierce, McLaughlin-Jenkins, Bertola-Harnad, Balogh-Bertola-Lee-McLaughlin-Prats Ferrer).

Given the proximity of interest, it is not surprising that new projects were started during the workshop, like the collaboration between Harnad and Wang on fermionic interpretation of matrix models with external source, the project (in advanced stage of completion by now) between Buckingham, Lee and Pierce on the Riemann–Hilbert approach of the self-avoiding random walkers with few outliers and the very active discussions on the algebro-geometric approach vs analytic one in studying the "loop equations" in the context of random matrix models, between Prats Ferrer, McLaughlin and Ercolani.

As often happens during workshops, papers may not be written in full but the seeds of fruitful collaborations are sown.

The organizers are extremely thankful to the support staff at BIRS for facilitating an extremely successful, pleasant, and smooth-running workshop.

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