COHOMOGENEITY ONE MANIFOLDS WITH POSITIVE SECTIONAL CURVATURE

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1. Background

Since the simplest non-trivial Riemannian manifold is the round sphere of radius r and (constant) curvature $1/r^2$, it is only natural that manifolds with positive curvature have played a central role since the beginning of global Riemannien geometry. In the (complete) non-compact case a theorem of Gromoll and Meyer (1970) asserts that the manifold is diffeomorphic to euclidean space, and in the compact case the classical Bonnet-Myers theorem (1932) imply that the fundamental group is finite. In even dimensions a result of Synge (1934) shows that the manifold is simply connected if it is orientable. The main issue in the subject is therefore to understand (compact) simply connected manifolds with positive curvature. Except for special obstructions for spin manifolds (stemming already from positive scalar curvature) the only known obstruction is the Betti number theorem due to Gromov (1980) which even applies to non-negative curvature: It provides a bound on the total Betti number which depends only on the dimension.

Under natural additional conditions there are celebrated results that identify the manifold with the sphere or with one of the other rank one symmetric spaces, i.e., the complex and quaternionic projective spaces, or the Cayley plane, the so-called CROSS'es. It is remarkable that above dimension 24 these are the only known (simply connected) manifolds of positive curvature. Additional examples have appeared in dimensions 6, 7, 12, 13, and 24. These examples include a complete classification of positively curved homogeneous manifolds due to combined work of Berger (1960), Wallach (1970), Aloff-Wallach (1972), and Berard-Bergery (1975) (one in each of the dimensions 6, 12, 13 and 24, and infinitely many in dimension 7). Non-homogeneous examples have been found by Eschenburg (1978) in dimensions 6, 7, and by Bazaikin (1996) in dimension 13 (one in dimension 6 and infinitely many in the other dimensions). All these examples are so-called biquotients, i.e., quotients of a compact Lie group G by a subgroup of $G \times G$ acting on left and right on G.

To advance the theory at this point it seems imperative to find new examples, a task which notoriously is very difficult as indicated above. For simplicity, and since all known examples have a fairly large group of isometries, it seems natural to look for new examples with large symmetry group. Attempts to classify manifolds with positive curvature and large isometry group provide a framework for a systematic search for new examples. One of the natural measuments for the size of the isometry group is its *cohomegeneity*, i.e., the dimension of the orbit space. For example, having minimal cohomogeneity 0 means that the isometry group acts transitively on the manifold, i.e., it is homogeneous. In analogy with the case of homogeneous manifolds, Wilking recently showed that in any fixed cohomogeneity, sufficiently high dimensional manifolds of positive curvature are CROSSs (up to tangential homotopy equivalance).

2. Project Desription

The previous section should provide ample justification for a serious investigation of simply connected cohomogeneity one manifolds of positive curvature with a complete classification as the final goal.

It is well known that the orbit space M/G of a simply connected cohomogeneity one G-manifold M, is an interval whose end points correspond to two singular orbits $B_\pm = G/K_\pm$ of codimension at least two, and where each interior point is a principal orbit G/H of codimension one. Moreover, $S^{l_\pm} = K_\pm/H$ are normal spheres to $B_\pm$, and $M = G \times_{K_-} D^{l_-+1} \cup G \times_{K_+} D^{l_++1}$ is the union of tubular neighborhoods of these orbits. In particular, M is determined by the subgroups $H \subset \{K_\pm\} \subset G$ and vice versa. We point out that in general a cohomogeneity one manifold can support many inequivalent cohomogeneity one actions. In particular there are many (linear) cohomogeneity one actions on spheres and more generally on CROSSes.

A remarkable first step towards the classification of cohomogeneity one manifolds of positive curvature is the recent result of L. Verdiani asserting that in *even dimensions* only CROSSes appear with their standard actions. - The same is false in odd dimensions. In fact, one observes that specific infinite subfamilies of the Eschenburg spaces and of the Bazaikin spaces are indeed of cohomogeneity one, as are three non CROSS normal homogeneous spaces, $B^7 = \mathrm{SO}(5)/\mathrm{SO}(3), W^7 = \mathrm{SU}(3)\,\mathrm{SO}(3)/\mathrm{U}(2)$, and $B^{13} = \mathrm{SU}(5)/\mathrm{Sp}(2)S^1$.

When specialising Wilkings theorem to the case of cohomogeneity one, the dimension beyond which all manifolds are like a CROSS is 72. In recent work by Grove, Verdiani, Wilking and Ziller it was shown that no cohomogeneity one exotic sphere (Kervaire sphere) supports an invariant metric with nonnegative curvature, and neither does any non-linear cohomogeneity one action on a standard sphere. In particular, if a positively curved cohomogeneity one manifold is homotopy equivalent to a CROSS, it is indeed a CROSS with a standard action. It thus remains to classify (simply connected) positively curved cohomogeneity one spaces below dimension 72, that are not homotopy equivalent to a CROSS.

The classification has two natural parts:

- Find obstructions on the manifold M, i.e., on $H \subset \{K_\pm\} \subset G$, due to positive curvature
- Find positively curved metrics on unobstructed manifolds M

Although we now believe that we have settled the first part, we cannot be sure until the second part has been carried out as well. It is stricking that the obstruction we have found in particular imply that the above bound of 72 can be replaced by 13. In fact

Theorem A. *Let M be a simply connected compact positively curved manifold on which a Lie group G acts isometrically with one dimensional orbit space. Then one has the following possibilities:*

(a) *M is eqivariantly diffeomorphisc to a rank one symmetric space with a linear cohomogeneity one action (all classified by Hsiang-Lawson).*
(b) *M is equivariantly diffeomorphic to a 7 dimensional positively curved Eschenburg space or 13 dimensional Bazaikin space with their natural isometric cohomogeneity one action.*
(c) *M is a 7 dimensional manifold on which $S^3 \times S^3$ acts isometrically by cohomogeneity one with finite isotropy group and singular orbits of codimension two.*

In parts (a) and (b) we already know the existence of positively curved metrics. In part (c) earlier work of the Grove and Ziller yield the existence of nonnegatively curved metrics. The existence of a positively curved metric is actually further significantly obstructed: If (p_-, q_-) and (p_+, q_+) are the slopes of the circles inside the singular isotropy groups K_-, K_+ as viewed in $S^3 \times S^3$, then either $H = \{\pm 1, \pm i, \pm j, \pm k\}$ or $H = \mathbb{Z}_2 \oplus \mathbb{Z}_4$. We denote the first family by $M_{(p_-,q_-),(p_+,q_+)}$ and the second one by $N_{(p_-,q_-),(p_+,q_+)}$. The only unobstructed manifolds left (at the moment) are then:

(i) $M_{(1,1),(1+2n,3+2n)}$
(ii) $N_{(1,1),(1+2n,2+2n))}$
(iii) $M_{(1,3),(3,1)}$
(iv) $N_{(3,1),(1,2))}$

Here $M_{(3,1),(1,3)}$ is actually the positively curved Berger space $SO(5)/SO(3)_{max}$, and $M_{(1,1),(1,3)}$ is S^7 with its linear cohomogeneity one action by $SO(4)$ coming from the isotropy representation of the symmetric space $G_2/SO(4)$.

It is also remarkable that the candidates in (i) and (ii) agree precisely with the 3-Sasakian manifolds arising from Hitchin's examples of self dual Einstein orbifolds on S^4. They are therefore $SO(3)$ orbifold principal bundles over S^4 and we observed that the total space happens to be a smooth manifold and not an orbifold.

A further intriguing property of the family $M_{(1,1),(1+2n,3+2n)}$ is that they are 2-connected with $\pi_3 = \mathbb{Z}_r$ and $r = (p_-^2 q_+^2 - p_+^2 q_-^2)/8 = n+1$ and hence are rational homology spheres. If they have a metric of positive sectional curvature, a theorem of Rong-Petrunin-Tuschmann would imply that the optimal pinching constant of any positively curved metric has to converge to 0 as r increases. This would be the first manifolds with such a phenomenon and would contradict a conjecture due to Fukaya.

Most of our work at BIRS was directed towards the construction of positively curved metrics on the manifolds described in (i) (and hence (ii)) above. We already knew that even this would be a formidable problem since the curvature formulas for invariant metrics are very complicated. Although our understanding and insights deepened siginificantly as a result of this process (see below), we do not yet know how to construct the desired metrics.

Here is a brief description of the key steps in our search. By invariance, the (most restricted) metrics in (i) on the principal orbits are given by metrics on three orthogonal two dimensional subspaces and hence by 3 symmetric 2×2 block, i.e., by 9 functions on the orbitspace interval. To define a smooth metric on the manifold, specific smoothness conditions at the boundary points where collapse occurs are imposed and completely understood for our candidates. To further simplify our investigations we have made repeated use of a well-known deformation of a G invariant metric first used by Berger for $G = R^1$ and in general by Cheeger. Basically, by schrinking the metric in the direction of the G orbits one generally gets more two planes with positive curvature. In our case we have determined exactly what it takes for a metric to have positive curvature modulo this so-called Berger-Cheeger trick. The full curvature operator for our examples splits into four symmetric 3×3 blocks with complicated expressions in terms of the 9 functions as entries. A sufficient but not quite necessary condition for positive curvature is that each of these blocks are positive definite. Since we already had explicit metrics with nonnegative curvature on our candidates it seemed like a natural attempt to deform these metrics. This turned out to be exceedingly hard, and although we did not prove it, we derived evidence that in fact there are no positively curved metrics on our candidates near the rigid nonnegatively curved metrics constructed eralier. Another intriguing starting point is to use our observation that our candidates coincide with the 3-Sasakian manifolds arising from Hitchin's examples of self dual Einstein orbifolds on S^4. So far, we have been unable to determine if this approach will provide the desired metrics, but the direct proceedure does not work. It is stricking that among the large class of cohomogeneity one $S^3 \times S^3$ manifolds with nonnegative curvature, our candidates are the only ones that even near the singular orbits admit positive curvature. Our most promissing approach so far, and the one we initiated at BIRS towards the end of our stay, is to start with metrics of positive curvature near the singular orbits, and attempt to match them at the boundary of tubular neighborhoods in a convex fashion. To make this work, it is however clear that much work and new ideas are needed.

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