# Applicable Harmonic Analysis

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The stated goal of the conference was to build on the great success of applicable harmonic analysis in the last decade by bringing together first-rate senior experts along with promising young researchers in the theory and application of wavelet analysis, non-linear approximation, computational fluid dynamics and other applications with an emphasis on a combination of theoretical development with practical applications. The mix of topics worked out marvelously. Although talks were originally grouped into sessions according to topic, the resolution of various scheduling conflicts mixed up these groupings quite thoroughly. However, since all talks were plenary and well-attended, the mixing did not prevent anyone from hearing any talk. It also made the workshop quite lively at times, with the participants sometimes gaining unexpected inspiration from superficially unconnected but in fact surprisingly relevant adjacent talks.

## 1 Wavelet Frames

Frames were introduced by R. J. Duffin and A. C. Schaeffer in 1952, in the context of non-harmonic Fourier series. But it was the context of multiresolution analysis and the wavelet expansion in applied harmonic analysis that has thrust frames and other means of generating redundant systems into the forefront. Frames are a system of functions that mimic many of the properties of orthogonal systems in the sense that the discrete norm of the coefficient sequence in a frame expansion of a function is equivalent to the norm of the function. However, frames may not be a basis, even when the frame bounds are tight. In other words, frames are often redundant systems. The added redundancy in representation would allow greater flexibility to achieve better approximation or isolation of desired features.

Ingrid Daubechies is famous with her construction of a family of compactly supported orthogonal wavelets with arbitrary smoothness. She is also a pioneering researcher in the area of frames. An important chapter was devoted to frames in her book *Ten Lectures on Wavelets* [3]. She was not able to participate in the workshop. But her co-workers, Bin Han, Amos Ron, and Zuowei Shen all gave remarkable talks on frames at the workshop. Their works represent the forefront of the current research in frames. In addition, Marcin Bownik, Guido Janssen, Qun Mo, and Morten Nielsen discussed their interesting results on the study of frames.

Amos Ron and Zuowei Shen applied shift-invariant systems to both Gabor frames and affine wavelets frames. Consequently, they established a unitary extension principle to construct tight frame systems. At this workshop, Amos Ron gave a talk about his latest joint work with Shen on generalized shift-invariant systems. They provided characterizations of the Bessel property and the frame property in terms of the norms and inverse norms of related matrices. It is expected that their results will have a significant impact on future research in this direction. Guido Janssen in his talk reported the progress he made in his study of Gabor frames. In particular, he emphasized the essential role played by the Ron-Shen criterion for deciding whether a Gabor system is a Gabor frame. Morten Nielsen devoted his talk to nonlinear approximation with tight wavelet frames. He gave characterizations of classical function spaces in terms of the coefficients in the expansion of tight wavelet frames. He also extended the well-known theory of nonlinear approximation of Ron DeVore from orthogonal wavelets to tight wavelet frames.

Zuowei Shen gave an exciting talk on applications of wavelet frames to image processing. He investigated high-resolution image reconstruction, which refers to the reconstruction of high-resolution images from multiple low-resolution, shifted, degraded samples of a true image. In order to achieve the goal of satisfactory reconstruction, he proposed an iterative algorithm based on wavelet analysis. He demonstrated the advantage of redundant systems (wavelet frames) over wavelet bases for the performance of the iterative algorithm. His numerical results showed that the reconstructed images from his wavelet algorithm were better than that obtained from the existent algorithms.

The work of Bin Han was concentrated on wavelet frames. During his visit to Princeton University, Bin Han together with Ingrid Daubechies completed a series of important papers on constructions of wavelet frames. The ideas initiated in those papers were developed further by Bin Han and Qun Mo in their talks. Han proposed several methods to construct dual wavelet frames and tight wavelet frames from refinable functions. He also generalized his methods to refinable vectors of functions and multivariate refinable functions. Starting from two symmetric generators Qun Mo discussed a way to construct tight wavelet frames with the desirable properties such as symmetry and high order of vanishing moments.

Marcin Bownik in his talk presented some recent results on canonical duals of wavelet frames, a topic investigated by Daubechies and Han earlier. He connected the property of shift-invariance with the existence of affine dual frames. Furthermore, he answered affirmatively a conjecture of Daubechies and Han about the period of a dyadic frame wavelet.

The subject of frames is currently a very active research area within the community of applicable harmonic analysis and has captured the attention and interest of many researchers in several related disciplines. The Journal *Applicable and Computational Harmonic Analysis* (ACHA) with C. K. Chui, R. R. Coifman, and I. Daubechies as editors-in-chief is planning to publish a special issue on this area, compiling the ongoing explosive research work and disseminating the important findings to the wide spectrum of ACHA readership. The special issue will be edited by C. Heil. R. Q. Jia, and J. Stoeckler. Some of the results presented in this workshop will be included in the special issue.

## 2 Spline Wavelets and Subdivision Schemes

Splines (piecewise polynomials) were introduced by I. J. Schoenberg in 1946 as efficient tools for interpolation and approximation. As the finite element method emerged, splines were soon applied to numerical solutions of differential equations. Being good representations of curves and surfaces, splines were also applied to computer-aided geometric design. The book *A Practical Guide to Splines* by Carl de Boor, [1], provided a guidance for wide applications of splines. In 1990, motivated by rapid development of wavelet analysis, Charles Chui and Jianzhong Wang initiated their study of spline wavelets. Their research and many other important topics were summarized in Chui's book *An Introduction to Wavelets*, [2].

At this workshop, Charles Chui gave a very interesting talk based on his joint work with Qing Tang Jiang on surface subdivision schemes generated by refinable bivariate spline function vectors. They introduced a direct approach for generating local averaging rules for both the $\sqrt{3}$ and 1-to-4 vector subdivision schemes for computer-aided design of smooth surfaces. The innovation of their work was to directly construct refinable bivariate spline function vectors with minimum supports and highest approximation orders and then to compute their refinement masks. The talk of Serge Dubuc was also related to vector refinement equations. In 1987, starting from cardinal splines, Serge Dubuc and Gilles Deslauriers obtained a family of compactly supported interpolatory refinable functions with arbitrary smoothness. It turned out their family of interpolatory refinable functions was closely related to the Daubechies' orthogonal refinable functions and wavelets. At this workshop, Serge Dubuc reported his recent work on Hermite subdivision schemes. His approach was to transform the initial scheme into a uniform stationary vector subdivision scheme. By investigating spectral properties of related matrices, he gave necessary and sufficient conditions for convergence of Hermite subdivision schemes.

Jianzhong Wang introduced some new multi-order spline wavelets in his approach to edge detection in images. Many edge detection methods use multi-scale representation of images in edge detection. His representation uses multi-scales derived from B-splines of different order. He discussed the advantages of the fast multi-order spline wavelet transform, particularly in dealing with edge detection in the presence of noise.

G. Donovan, J. S. Geronimo, D. P. Hardin, and P. R. Massopust pioneered the development of spline wavelets into multiple wavelets. In particular, by using fractal interpolation functions, they constructed compactly supported orthogonal spline multi-wavelets. At this workshop, Doug Hardin presented a method for generating local orthogonal bases on arbitrary univariate grids via a squeeze map. He also considered wavelets on semi-regular grids in the plane. Another interesting talk on spline wavelets was given by Song-Tau Liu. He reported his joint work with Rong-Qing Jia on wavelet bases of Hermite cubic splines on the interval. On the basis of Hermite cubic splines, they explicitly constructed a pair of spline wavelets which are continuously differentiable and supported on the interval $[-1, 1]$. These wavelets were then adapted to the interval with a very simple construction of boundary wavelets. The wavelet basis was proved to be globally stable and applied to numerical solution of the Sturm-Liouville differential equation with two-point boundary value conditions. Numerical examples were presented to demonstrate the advantage of the wavelet basis.

Concerning spline wavelets and subdivision schemes there are still many open problems that require further investigations. A main issue is how to construct globally stable wavelet bases with desirable properties on arbitrary triangulations. A related problem is subdivision schemes on irregular meshes in high dimensional spaces. These challenging problems would require creative ideas and innovative techniques.

## 3 Fractal Geometry and Tiling

Ka-Sing Lau and Yang Wang, two prominent mathematicians in the area of fractal geometry, gave stimulating talks at this workshop. A main issue in fractal geometry is the self-affine tiling. A self-affine pair consists of an expansive matrix with integer entries and a digit set. Such a pair induces a set called the attractor. In fact, the characteristic function of the set induced is the normalized solution of the refinement equation associated with the expansive matrix and the digit set. The question is to characterize the digit set such that the corresponding attractor tiles the underlying Euclidean space by translations. The Cantor set is a well-known self-similar tile. Both Ka-Sing Lau and Yang Wang have done extensive research on this subject. A notable achievement was the use of number theory by J. C. Lagrias and Y. Wang in their study of self-similar tilings. At this workshop, Ka-Sing Lau presented some new results on characterizations of the digit sets with the determinants of the expansive matrices being prime numbers. In particular, he and X. G. He together solved a conjecture of Lagarias and Wang. The talk of Yang Wang connected the theory of tiling and wavelets to spectral measures, which allow orthonormal bases consisting of complex exponentials. His work pointed a new direction in applicable harmonic analysis. Both Ka-Sing Lau and Yang Wang posed some interesting open problems.

## 4 Computational and Applied Harmonic Analysis

A key area of applied harmonic analysis is the solution of equations describing fluids. The theoretical school was represented at our workshop by senior researchers Susan Friedlander and Marco Cannone, who discussed their existence and uniqueness results for a dyadic version of Euler's equation and for the mild version of the Navier-Stokes equations, respectively. Cannone's talk at the start of the June 9 sessions was especially helpful to younger researchers, as it began with a clear introduction to Littlewood-Paley decompositions and their use in his powerful existence theory. Friedlander also pitched the first part of her talk to non-experts, describing a modified Euler equation in which derivatives were replaced by sparse matrices acting on Haar wavelets in a similar way. She described initial conditions that evolve to blow up in finite time. Participants thus saw two very different results, existence and blow-up, derived from different simplifications using the same tool from harmonic analysis.

The numerical analysis school of fluid dynamics was more broadly represented. Marie Farge was one of the first researchers to use wavelet decompositions, to enhance numerical simulations of the Navier-Stokes equation, applying the methods in Grossmann and Morlet's seminal 1984 paper. Her talk described a preliminary result on the segmentation of computed vorticity fields using wavelet thresholding. This technique is used in spatially adaptive numerical simulations, as the (large) segments containing no coherent vortices may be approximated more crudely in order to save computing time. Younger researchers Nicholas Kevlahan and Kai Schneider, on the other hand, described wavelet-based feature detection methods for deciding when to refine the grid in order to maintain high accuracy in a Navier-Stokes simulation.

Senior investigator Wolfgang Dahmen then gave a nice survey of his long-running general research program on the multiscale decomposition of linear and nonlinear operators, which has produced results of interest to both the theoreticians and numericists. This time the application was to existence of solutions in the calculus of variations, and their numerical computation. This may have inspired the theoreticians to simplify their model equations into variational form, and the numericists to try some new iterative methods. As if to drive home the point, Yuesheng Xu described how multiscale operator decomposition can be used to solve general second-kind integral equations numerically by iterative refinement from coarse grids.

Thus the talks presented by this very diverse group of theoretical and numerical analysts formed a remarkably coherent collage of the applications of multiscale and wavelet decompositions. Lin Wei, who would have contributed his results on the elasticity equation to this collage, unfortunately could not attend because of visa difficulties.

For the June 11 afternoon session, a number of people were invited to speak on particularly successful applications of harmonic analysis and wavelets, with the idea being to widen everyone's awareness. Image processing is one such application. Peter Oswald gave a survey of wavelet decomposition methods for computer graphics rendering of surfaces in 3-D. Richard Baraniuk described a spatially adaptive algorithm for image compression that made local decisions on whether to encode edges or textures in a small region based on wavelet components. Jacques Liandrat's results hit the same target from a different direction, using new parameters extracted from the "lifting," or predict-and-update implementation of the wavelet transform to make spatially-adaptive choices of local wavelet transform for compact image coding.

Time series analysis is another area where wavelet transforms are a standard tool. Metin Akay described how to detect obstructive sleep apnea from electroencephalograms using filtering in the wavelet domain. Sinan Gunturk took the opposite tack and spoke on best approximation of time series by filtered pulse trains. These two talks were among the most entertaining of the workshop: Akay's because he was working on snoring, and Gunturk because of his amusing analogy with the "fair duel" problem.

# 5 Non-linear Approximation

Non-linear approximation has played an ever increasing important role applied harmonic analysis and other areas. It has been particularly important as a means to choose best approximations using a specified number of terms, $n$-term approximations, from wavelet and frame representations of functions when the functions are believed to belong to some smoothness spaces, such as the Besov spaces. Two of the most prominent researchers in non-linear approximation, Ron DeVore and Vladmir Temlyakov, presented aspects of their work which were complemented by the talks from younger collaborators.

Ron DeVore's talk on adaptive numerical methods for pde's could well have also been place in the last section, showing some of the natural overlap that occurred. The talk centered around the basic Poisson problem

$$-\nabla u = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega,$$

where $\Omega$ is a polygonal domain in $\mathbb{R}^2$ and $\partial\Omega$ is its boundary. Though adaptive methods have been successfully applied to solve such problems, it was only recently that the methods were shown to converge. There had been no analysis on the rate of convergence. DeVore in collaboration with Cohen and Dahmen had recently developed an algorithm with optimal efficiency for adaptive wavelet methods using $n$-term approximation. For adaptive finite element methods, there was no known algorithm with a proven rate of convergence except in the univariate case. The talk presented work of Ron DeVore, Peter Binev, and Wolfgang Dahmen that provides such a method. The method is not that much different than others based on bulk chasing of a-posteriori estimated except that it introduces a coarsening step. The main ideas of the coarsening step were presented in the talk by Peter Binev.

Best basis selection in greedy algorithms was the basis of the talks of Vladmir Temlyakov and Guergana Petrova. When considering the expansion of functions in terms of basis elements for best $n$-term approximation there are a variety of settings. A wavelets basis gives a unique expansion, frames allow some redundancy and therefore some additional flexibility, but one can consider a much more general problem of choosing at each step a basis element from a dictionary and its coefficient. Temlyakov takes a *greedy* approach from non-linear approximation in his selection of next basis element, and introduces a new selection of the coefficients. He is about to prove convergence in any uniformly smooth Banach space.

Guergana Petrova in her talk considers the Banach space $L_p$ and considers the problem of approximation of a function class $\mathcal{F}$ in $L_p$. Here she will first choose a basis and then use $n$-term approximation with the elements from that basis. An additional degree of non-linearity is added to the problem by allowing the choice of basis to depend on the function class. This requires the determination of what would be a best basis for a particular function class as characterized by intrinsic properties of that function class. David Donoho first studied this problem in Hilbert space ($L_2$) with the competition taken over all complete orthonormal systems. To extent these ideas beyond $L_2$ to $L_p$ requires a replacement for the complete orthonormal systems. The proper setting is to use *greedy bases*, which have been characterized by Konyagin and Temlyakov as democratic unconditional bases. In this setting the problem can be successfully handled. The techniques involve metric entropy an encoding and can be applied in a variety of settings.

# 6 Sampling Theory and Hyperfunctions

Recently there has been a lot of interest generated by Statistical Learning Theory, especially as a potential tool in bioinformatics. Learning theory studies objects by looking at random samples. The main question that Ding-Xuan Zhou dealt with in his lecture is to determine the number of samples needed to ensure an error bound with a certain level of confidence. These problems often involve a kernel and the study of its associated reproducing kernel Hilbert space. There is a natural connection to approximation theory, harmonic analysis, and wavelets. This interesting talk introduced the audience to the use of multiresolution analysis, wavelets, and sampling theory to the study of problems of learning through support vector machines. This provided some insight into a rapidly developing field in which topics central to the theme of this Workshop could have major implications.

In another direction, Kurt Jetter looked at a generalization of Shannon sampling, the so-called polyphase sampling. Instead of sampling data at the integers using integer translates of fixed function, the polyphase operator uses the translates of a finite set of functions to sample at different phase shifts of the data. The talk set out various approximation properties of polyphase operators and linked this notion to concepts in wavelet theory.

In an entirely different vein, Dohan Kim gave a very interesting lecture on the connections of some fundamental concepts in Fourier analysis, comparing results in hyperfunctions to the Schwartz theory of distributions. The talk was interesting both for its mathematical content and the historical perspective.

# References

[1] Carl de Boor, *A Practical Guide to Splines, Revised Edition*, Springer-Verlag, New York, 2001.

[2] Charles K. Chui, *An Introduction to Wavelets*, Academic Press, San Diego, 1992.

[3] I. Daubechies, *Ten Lectures on Wavelets*, volume 61 of CBMS Conference Series in Applied Mathematics, SIAM, Philadelphia, 1992.