Equilibrium and non-equilibrium pattern formation in soft matter: From elastic solids to complex fluids (22w5127)

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1 Overview of the Workshop

Nature is full of patterns that emerge spontaneously from featureless environments. From microscopic snowflakes to coral reefs and large-scale river networks, pattern formation leads to systems of extraordinary intricacy and beauty. Examples exist across orders of magnitude in size, and include many biological and technologically relevant assemblies. This workshop will bring together distinct communities that investigate pattern formation both in and out of equilibrium, with the aim that this cross-pollination will lead to new insights that advance the field.

Many pattern forming systems are intrinsically non-equilibrium, either being constantly driven away from any stable equilibrium, or getting arrested while on their way to a steady state. Examples include turbulent fluid flows as well as the dendritic growth at snowflake tips. The growth of such structures is often nonlinear, and their mathematical description involves the solution of time-dependent partial differential equations.

Other times patterns form in equilibrium, as local or global minimizers of some non-convex energy functional. Examples include the orderly labyrinths of wrinkles that emerge when thin elastic films are squeezed, and the striped or branched domains seen in micromagnetics. For such "energy-driven pattern formation", a focus on determining the scaling law of the minimum energy in the relevant parameters has yielded insights. This approach continues to provide a basis for charting new territory alongside novel experimental discoveries.

Progress on these issues presently occurs largely in two distinct fields, with practitioners working either on the time-dependent, non-equilibrium aspects of pattern formation (mainly in fluid dynamics) or on the static, equilibrium aspects (mainly in materials science and solid mechanics). Although the mathematical descriptions of these systems are indeed different — the former involves dynamical systems while the latter involves energy minimization — some connections are beginning to emerge. This BIRS workshop brought together a diverse audience of mathematicians, physicists, and engineers working on the fundamental aspects of pattern growth, both in and out of equilibrium. The meeting stimulated a broad and lively discussion of theoretical developments, analytical and computational methodologies, and applications to device design.

2 Recent Developments and Workshop Objectives

The study of pattern formation is currently situated at a variety of intersections. While our understanding of the fundamental principles leading to complex structures continues to grow, we have already started to exploit their spontaneous growth to create novel materials and fabrication processes. Mechanical instabilities, for example, long associated with the idea of failure and seen as needing to be suppressed or avoided, are now thought of as opportunities to build potentially useful materials and structures. Not only is the field of pattern formation in soft materials and complex fluids broad and multifaceted, but so are the interdisciplinary communities involved in the research. The question of how such patterns are formed is at the intersection of several fields including mathematics, theoretical mechanics, physics, biology, geology, and computer science, and to make progress we believe it is vital to bring these communities together. This BIRS workshop contributed to this ambitious goal.

Mathematical background. Mathematically speaking, the patterns being discussed are governed by systems of nonlinear partial differential equations arising from a balance of forces. Though the exact form of the differential equations to be solved depends on the problem at hand, some common features of their analysis persist. In the time-dependent, non-equilibrium case, conservation laws (such as of mass or energy) play a key role. When these can be derived from the equations they often permit the application of functional inequalities implying the existence and uniqueness of solutions, and governing the features of the patterns they describe. In a possibly unexpected but directly analogous way, the very same functional inequalities have now found their use in the mathematical analysis of equilibrium patterns, where the questions of determining optimal scaling laws and optimal constants are tied up with describing the fine details of the patterns themselves.

Single versus multiple length scales. Fractal-like patterns display complexity on a wide range of scales, and are an established subject of study. Less obvious is the question of whether self-similar cascades could offer a link between equilibrium patterns and non-equilibrium patterns. One promising example comes from a recent study on optimal heat transport, which carried over methods originally developed in the study of equilibrium patterns to construct a fluid flow achieving optimal wall-to-wall transport of heat by an incompressible fluid. The structure is a self-similar branched flow pattern, which mimics the form of wrinkle cascades observed in experiments on thin elastic sheets. Not only are the patterns similar at first glance, but the functional analysis also exhibits a curious link. Could similar connections be exploited to bring tools from equilibrium to non-equilibrium settings in other problems? Cascades may also exist in time rather than in spatial dimensions, such as in dynamic coarsening. On the other hand, some patterns contain only a single lengthscale, for instance a uniform wrinkle wavelength. Even in this case, other nontrivial lengthscales can arise due to geometric constraints that frustrate the formation of parallel wrinkles, which is an active area of study.

Orderly and disorderly patterns and shape change. Equilibrium buckling patterns on the surfaces of elastic media often display a high degree of spatial order, whereas a crumpled paper sheet has a more stochastic appearance as it gets stuck in one of many local energy minima. In the non-equilibrium setting, one finds examples of transitions from disordered to ordered growth, for instance in a number of interfacial patterns that grow into fractal or highly branched patterns under isotropic conditions but transition into regular dendritic structures in the presence of anisotropy. Are there general principles for determining whether a pattern forming system is capable of exhibiting order and/or disorder? Are these concepts linked to the ability of the material to sustain different types of bulk shape change?

Failure versus flow. The approach to describe the formation of patterns faces challenges when the behavior of a system can switch between fluid-like or solid-like. An example of a physical system that displays this behavior is a dense granular suspension, which forms smooth fingers in its fluid state, but can, as a result of applied stress, reversibly switch to a solid-like state where fracture patterns are formed. Although always out of equilibrium, a single method to explain or predict emerging patterns does not yet exist. In particular, the traditional methods that are based on a fluid dynamics approach can only partially describe the observed patterns in such systems. Are approaches from equilibrium pattern formation, or more generally from solid mechanics, able to advance our understanding of these complex systems?

3 Presentation Highlights

A central question of the workshop was whether pattern formation could be controlled to generate a desired structure in space and time. There is an apparent paradox in this assertion, which is due to the general self-selection of patterns. Pattern formation is often investigated when the structures are formed spontaneously, and therefore any length scales are self-selected without the need of any control. Trying to gain control over the pattern formation therefore appears to remove this most interesting aspect of spontaneous formation.

In fluids, length scales and shapes are typically set by instabilities that depend on fluid properties, e.g., surface tension, viscosity or density, along with any driving mechanisms such as a pressure gradient, velocity, or gravity. During the workshop, several possibilities on how to influence these instabilities were presented and discussed. Strategies for an increased level of control were identified, while still exploiting the self-driving behavior of the fluid instabilities to form patterns.

One method of control that was discussed was to pre-pattern a solid or viscous substrate to provide the fluid with a template. If the length scale of the template is close enough to the natural length scale of the instability, then the pattern will adjust to the template. This behavior was demonstrated for Rayleigh-Taylor and Rayleigh-Plateau instabilities [2]. The instability parameters can also be influenced more directly, as was demonstrated for the Rayleigh-Taylor instability that depends on a balance between surface tension and gravitational acceleration. While surface tension is notoriously difficult to change by any significant amount, the effective gravitational acceleration can be adjusted using centrifugal acceleration. This can be used to change both the length scale of the instability and the shape of the fluid deformation [13].

Another way that was discussed of tuning pattern growth in fluid instabilities is to exploit the role of diffusion in the viscous fingering instability, in the case that the displacing and the displaced fluids are miscible. As diffusive effects become comparable to advective effects from the pressure-driven flow, the instability can be suppressed. Surprisingly though, for even lower Péclet numbers, the presence of diffusion turned out to allow for a different type of instability to form where it is otherwise forbidden.

Patterns are also used to program macroscopic shape changes in solid materials. In this direction, a few presentations discussed control and design of mechanical metamaterials, focusing on the example of kirigami, or cut elastic sheets. These talks emphasized a strong connection between physical modeling and rigorous mathematical analysis, where functional analytic bounds were shown to be sharp in experiments and simulations. Problems discussed included describing the bulk or homogenized response of the "rotating squares" lattice and its parallelogram variants [19, 20, 14], as well as the bulk behavior of the kagome lattice [11, 9]. These talks studied periodic kirigami sheets in the plane. A Gamma-convergence based framework was discussed, and the talks identified the appropriate energy scales and effective variables. A separate question was about the ability of a general, non-periodic kirigami design to deploy out of the plane into a configuration at the boundary of the admissible set [3, 8].

Finally, one presentation sought to draw connections and analogies between pattern formation in solids and fluids, to further our understanding on both sides [4]. Wrinkles have conventionally been studied in solid films, but there is recent excitement in the field surrounding the dynamic wrinkling of viscous bubbles during their rupture. This presentation applied concepts from geometry and tension-field theory to the physics of fluid films, to seek a unified understanding of this fluid wrinkling phenomenon, and envisioned a broader class of fluid films that could be studied or controlled in this manner, by harnessing the effects of the film curvature.

4 Outcomes and Participant Feedback

This workshop allowed participants from different disciplines and areas of expertise to come together and learn from one another. The broad nature of the workshop was appreciated by the participants who were exposed to various topics across equilibrium and out-of-equilibrium modeling. One participant noted the difference in the scientific approach between a mathematician and a physicist and the benefit of having different viewpoints being represented in the workshop. At the same time, some noted the challenges in following

Figure 1: Virtual Group Photo

Figure 2: In-person Group Photo

highly technical talks in the area they were not experts in and suggested having some introductory lectures or tutorials to make the talks more accessible in a future iteration.

Another successful outcome of the workshop was the informal interactions between the participants that happened organically. Particularly, in-person participants appreciated informal discussions across disciplines with those they had not previously met or had not seen in a long time. At the same time, virtual participants universally appreciated the hybrid nature of the workshop, which allowed them to participate remotely, including the opportunity to give talks in some cases. Even those who did not give talks were able to maintain a strong presence at the workshop through their online participation, which was also appreciated by those who were there in person.

Some participants noted a few IT issues that hindered the smooth integration of virtual and in-person participation, including the inability to share one's screen during breakout sessions. There were also some technical difficulties in setting up the talk for the hybrid audience, which could be improved. But, overall, the workshop was successful at blending the online and in-person approaches.

5 Participants List

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References

[1] K. A. Bacik, B. S. Bacik, and T. Rogers, Lane formation in complex active flows, *submitted* (2022).

[2] L. Cai, J. Marthelot, and P. T. Brun, Instability mediated self-templating of drop crystals, *Science Advances* **8** (2022), eabq0828.

[3] G. Chaudhary, L. Niu, M. Lewicka, Q. Han, and L. Mahadevan, Geometric mechanics of random kirigami, *submitted* (2022).

[4] B. Davidovitch and A. Klein, How viscous bubbles collapse: topological and symmetry-breaking instabilities in curvature-driven hydrodynamics, *submitted* (2022).

[5] D. Duffy, L. Cmok, J. S. Biggins, A. Krishna, C. D. Modes, M. K. Abdelrahman, M. Javed, T. H. Ware, F. Feng, and M. Warner, Shape programming lines of concentrated Gaussian curvature, *Journal of Applied Physics* **129** (2021), 224701.

[6] F. Feng, M. Barnes, K. Dradrach, D. Duffy, and J. S. Biggins, Geometry and mechanics of intrinsically curved creases, *in preparation* (2022).

[7] F. Feng, D. Duffy, M. Warner, and J. S. Biggins, Interfacial metric mechanics: stitching patterns of shape change in active sheets, *Proceedings of the Royal Society A* **478** (2022), 20220230.

[8] Q. Han, M. Lewicka and L. Mahadevan, Isometric immersions with rectifiable geodesics, *submitted* (2022).

[9] R. V. Kohn and X. Li, The macroscopic elastic energy of the Kagome lattice metamaterial, *in preparation* (2022).

[10] M. L'Estimé, L. Duchemin, E. Reyssat, and J. Bico, Fingering instability in adhesion fronts, *under revision at Journal of Fluid Mechanics* (2022).

[11] X. Li and R. V. Kohn, Some results on the Guest-Hutchinson modes and periodic mechanisms of the Kagome lattice metamaterial, *submitted* (2022).

[12] J. Marthelot, B. Roman, J. Bico, J. Teisseire, D. Dalmas, and F. Melo, Self-Replicating Cracks: A Collaborative Fracture Mode in Thin Films, *Physical Review Letters* **113** (2014), 085502.

[13] J. Marthelot, E. F. Strong, P. M. Reis, and P. T. Brun, Designing soft materials with interfacial instabilities in liquid films, *Nature Communications* **9** (2018), 4477.

[14] P. Plucinsky and I. Tobasco, Rigidity and energy scaling laws for kirigami in the plane, *in preparation* (2022).

[15] P. Shah, S. Arora, and M. M. Driscoll, Coexistence of solid and liquid phases in shear jammed colloidal drops, *submitted* (2021).

[16] E. Siéfert, E. Reyssat, J. Bico, and B. Roman, Programming stiff inflatable shells from planar patterned fabrics, *Soft Matter* **16** (2020), 7898–7903.

[17] T. E. Videbæk, Delayed onset and the transition to late time growth in viscous fingering, *Physical Review Fluids* **5** (2022), 123901.

[18] T. E. Videbæk and S. R. Nagel, Diffusion-driven transition between two regimes of viscous fingering, *Physical Review Fluids* **4** (2019), 033902.

[19] Y. Zheng, I. Niloy, P. Celli, I. Tobasco, and P. Plucinsky, Continuum field theory for the deformations of planar kirigami, *Physical Review Letters* **128** (2022) 208003.

[20] Y. Zheng, I. Tobasco, P. Celli, and P. Plucinsky, Modeling planar kirigami metamaterials as generalized elastic continua, *submitted* (2022).