Isoperimetric properties of spaces with nonnegative Ricci (or nonnegative scalar) curvature

Gioacchino Antonelli

Courant Institute (NYU)
ga2434@nyu.edu

Recent advances in comparison geometry, Hangzhou, China

Based on joint works with E. Bruè, M. Fogagnolo, F. Glaudo, E. Pasqualetto, M. Pozzetta, S. Nardulli, D. Semola

February 28, 2024

The isoperimetric profile

The isoperimetric profile

Let M be a Riemannian manifold with volume measure vol and perimeter measure Per.
For $v \in (0, \mathrm{vol}(M))$, the **isoperimetric profile** function I_M at v is

$$I_M(v) := \inf\{\mathrm{Per}(F) : \mathrm{vol}(F) = v\}.$$

If $I_M(\mathrm{vol}(E)) = \mathrm{Per}(E)$ we say that E is an **isoperimetric set**.

The isoperimetric profile

The isoperimetric profile

Let M be a Riemannian manifold with volume measure vol and perimeter measure Per.
For $v \in (0, \mathrm{vol}(M))$, the **isoperimetric profile** function I_M at v is

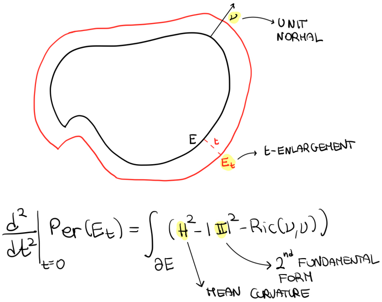
$$I_M(v) := \inf\{\mathrm{Per}(F) : \mathrm{vol}(F) = v\}.$$

If $I_M(\mathrm{vol}(E)) = \mathrm{Per}(E)$ we say that E is an **isoperimetric set**.

$\star$ Balls are (unique) isoperimetric sets in $\mathbb{R}^n$. Thus, $I_{\mathbb{R}^n}(v) = n(\omega_n)^{\frac{1}{n}} v^{\frac{n-1}{n}}$;

$\star$ The definition makes sense also in metric measure spaces [Ambrosio, '02, Adv. Math.], [Miranda, '03, JMPA].

Ricci curvature (bounded below) affects isoperimetry



Theorem (Lévy–Gromov isoperimetric inequality, Gromov, '80, Preprint)
Let $n \geq 2$, and M be a smooth n-dimensional complete Riemannian manifold with $\mathrm{Ric} \geq n-1$. Hence, for every $t \in [0,1]$,

$$\frac{I_M(t \cdot \mathrm{vol}(M))}{\mathrm{vol}(M)} \geq \frac{I_{\mathbb{S}^n}(t \cdot \mathrm{vol}(\mathbb{S}^n))}{\mathrm{vol}(\mathbb{S}^n)}.$$

What about isoperimetric sets in noncompact spaces (with nonnegative curvature)?

Nonexistence can happen

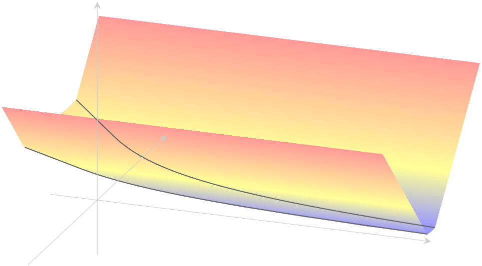
Theorem (A.–Glaudo, '23, Preprint)

For every $n \geq 3$ there is a smooth complete noncompact Riemannian manifold with $\mathrm{Sec} > 0$ that does not have isoperimetric sets with volume $v < 1$, while it does for $v > 1$.

⋆ Similarly for the relative isoperimetric problem in unbounded convex bodies in $\mathbb{R}^n$;

⋆ $n \geq 3$ is sharp due to [Ritoré, '02, JGEA]; $v > 1$ is sharp among examples with nondegenerate asymptotic cones [A.–Bruè–Fogagnolo–Pozzetta, '22, Calc. Var. PDE].

Sketch.

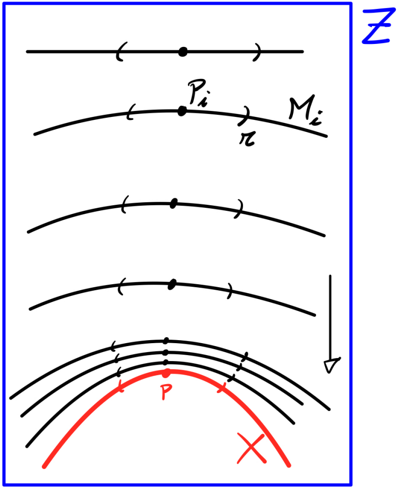


$\star$ Construct a convex body by cutting $\Sigma \times \mathbb{R}$, where $\Sigma \subseteq \mathbb{R}^n$ is a convex cone in the upper halfspace;

$\star$ Then, take the boundary and approximate it with smooth hypersurfaces.

Pointed Gromov–Hausdorff convergence (Gromov, '81)

$$(M_i, \mathrm{d}_i, p_i) \longrightarrow_{\mathrm{pGH}} (X, \mathrm{d}_X, p)$$



Ricci-limit spaces

Theorem (Gromov, '81, Pub. Math. IHES)

Let $K \in \mathbb{R}$, $n \in \mathbb{N}$. The class of smooth pointed n-dimensional complete Riemannian manifolds (M, p) with $\mathrm{Ric} \geq K$ is precompact in the pGH topology.

Ricci-limit spaces

Theorem (Gromov, '81, Pub. Math. IHES)

Let $K \in \mathbb{R}$, $n \in \mathbb{N}$. The class of smooth pointed n-dimensional complete Riemannian manifolds (M, p) with $\mathrm{Ric} \geq K$ is precompact in the pGH topology.

1. Any limit is called Ricci-limit space (RLS). Structure properties of RLS investigated in the seminal works [Cheeger–Colding, '96–'00, Ann. of Math., JDG].
2. Impulse to the study of geometry of CD and RCD spaces [Sturm, '06, Acta Math.], [Lott–Villani, '09, Ann. of Math.], [Ambrosio–Gigli–Savaré, '14, Duke Math. J. + Inv. Math.]...

When the isoperimetric set wants to escape at infinity

Theorem (A.–Nardulli–Pozzetta, '22, ESAIM:COCV; A.–Pasqualetto–Pozzetta–Semola, '22, ASENS)

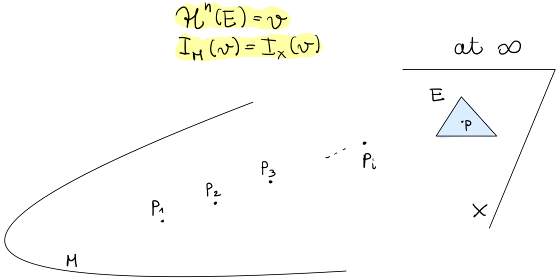
Let (M, d) be a smooth complete n-dimensional noncompact Riemannian manifold with $\mathrm{Ric} \geq 0$, and $\inf_{p \in M} \mathrm{vol}(B_1(p)) > 0$. Let $v \in (0, \mathrm{vol}(M))$. If there is no isoperimetric set of volume v in M, then the following holds. There exists a limit at infinity X, i.e.,

$$(M, \mathrm{d}, p_i) \longrightarrow_{\mathrm{pGH}} (X, \mathrm{d}_X, p) \quad \textit{for diverging } p_i,$$

with $I_X(v) = I_M(v)$, and there is an isoperimetric set of volume v in X.

Moral: Either you have an isoperimetric set in M or in one of its limits at infinity.

Sketch.



⋆ It is proved by **concentration-compactness**;

⋆ It holds for arbitrary Ricci lower bounds and in the **nonsmooth setting**.

Application: Sharp concavity of the isoperimetric profile on spaces with Ricci curvature bounded below

The concavity result

Theorem (A.–Pasqualetto–Pozzetta–Semola, '22, ASENS)

Let $n \geq 2$, and M be an n-dimensional complete Riemannian manifold with $\mathrm{Ric} \geq 0$. Then

$$I_M^{\frac{n}{n-1}} \text{ is concave.}$$

The concavity result

Theorem (A.–Pasqualetto–Pozzetta–Semola, '22, ASENS)

Let $n \geq 2$, and M be an n-dimensional complete Riemannian manifold with $\mathrm{Ric} \geq 0$. Then

$$I_M^{\frac{n}{n-1}} \text{ is concave.}$$

$\star$ In the **compact** case known from [Bavard–Pansu, '86, ASENS] ($n = 2$), and [PhD Thesis, '97, Bray], [Bayle, '04, IMRN].

Two ingredients: **existence of isoperimetric sets**, **second variation of the area on isoperimetric boundaries**;

The concavity result

Theorem (A.–Pasqualetto–Pozzetta–Semola, '22, ASENS)

Let $n \geq 2$, and M be an n-dimensional complete Riemannian manifold with $\mathrm{Ric} \geq 0$. Then

$$I_M^{\frac{n}{n-1}} \text{ is concave.}$$

⋆ In the **compact** case known from [Bavard–Pansu, '86, ASENS] ($n = 2$), and [PhD Thesis, '97, Bray], [Bayle, '04, IMRN].
Two ingredients: **existence of isoperimetric sets**, **second variation of the area on isoperimetric boundaries**;

⋆ *... It is not clear whether the [concavity] can hold outside a Riemannian setting...* [Ledoux, '11] referring to [Milman, '09, Inv. Math.]. Our result holds in the **nonsmooth** setting (e.g., Alexandrov spaces, RCD spaces), and for **arbitrary Ricci lower bounds**.

Quick overview of consequences of the concavity of $I^{\frac{n}{n-1}}$

⋆ Connectedness of isoperimetric regions;

⋆ Lipschitz continuity for the isoperimetric profile;

⋆ Uniform density estimates for isoperimetric sets;

⋆ Uniform diameter bounds for isoperimetric sets;

⋆ Stability of mean curvature under pointed Gromov–Hausdorff convergence;

⋆ A new proof of the sharp isoperimetric inequality in n-dimensional metric measure spaces with $\mathrm{Ric} \geq 0$ and nondegenerate asymptotic cones [A.–Pasqualetto–Pozzetta–Semola, '22, Math. Ann.]. First proved in [Agostiniani–Fogagnolo–Mazzieri, '20, Inv. Math.];

⋆ An alternative proof of the Lévy–Gromov isoperimetric inequality;

⋆ Small/large asymptotics and monotonicity of the isoperimetric profile...

How to survive without 2$^{\mathrm{nd}}$ variation and prove the concavity of the profile: Mean curvature barriers

Mean curvature in the nonsmooth setting

Theorem (A.–Pasqualetto–Pozzetta, '22, Nonlinear Anal.; A.–Pasqualetto–Pozzetta–Semola, '22, ASENS)

Let (X, d) be an n-dimensional RLS with $\mathrm{Ric} \geq 0$, and $E \subset X$ be an isoperimetric set. Then, for some $H \geq 0$, we have in the distributional sense

$$\Delta \mathrm{d}_{\overline{E}} \geq \frac{H}{1 + \frac{H}{n-1}\mathrm{d}_{\overline{E}}}, \quad \text{on } E$$

$$\Delta \mathrm{d}_{\overline{E}} \leq \frac{H}{1 + \frac{H}{n-1}\mathrm{d}_{\overline{E}}} \quad \text{on } X \setminus \overline{E}\,.$$

where $\mathrm{d}_{\overline{E}}$ is the signed (> 0 outside, < 0 inside) distance function from E.

$\star$ Encoding info on the mean curvature through Laplacian comparison has appeared in the smooth setting in [Wu, '79, Acta Math.], [Caffarelli–Cordoba, '93, Diff. Int. Equations].

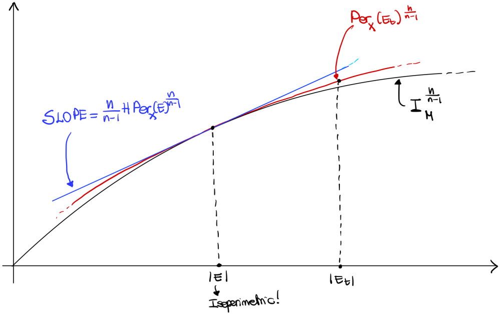
Concavity: producing the line touching above $I^{\frac{n}{n-1}}$

Proposition (A.–Glaudo, '23, Preprint, *and where else?*)

Let (X, d) be an n-dimensional RLS with $\mathrm{Ric} \geq 0$, and $E \subset X$ be an isoperimetric set. Then there is $H \geq 0$ for which

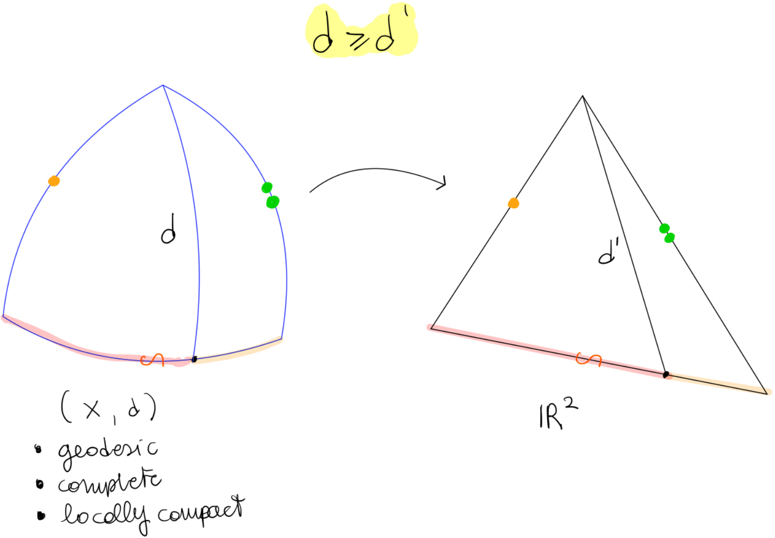
$$\mathbb{R} \ni t \mapsto \mathrm{Per}(E_t)^{\frac{n}{n-1}} - \frac{n}{n-1} H \mathrm{Per}(E)^{\frac{n}{n-1}} \mathrm{vol}(E_t),$$

achieves its maximum at $t = 0$.



Another application: When generalized existence is improved to existence

How to visualize *nonnegative curvature* (à la Alexandrov)



Existence for all volumes in 2D

Theorem (A.–Pozzetta, '23, Preprint)

Let (X, d) be a 2-dimensional nonnegatively curved metric space. Then isoperimetric sets exist for every volume.

For smooth surfaces [Ritoré, '02, JGEA].

Existence for all volumes in 2D

Theorem (A.–Pozzetta, '23, Preprint)

Let (X, d) *be a 2-dimensional nonnegatively curved metric space. Then isoperimetric sets exist for every volume.*

For smooth surfaces [Ritoré, '02, JGEA].

$\star$ **Sketch.** Use generalized existence. If the mass is lost at infinity:

Every limit at infinity splits as $\mathbb{R} \times Y$.

$\star$ Then models at infinity are: $\mathbb{R}^2$, or $\mathbb{R} \times [0, +\infty)$, or $\mathbb{R} \times \mathbb{S}^1(\rho)$, or $\mathbb{R} \times [0, \ell]$. In each case find *isoperimetrically more convenient* sets on the space.

Sharp isoperimetric inequality: maximal volume growth

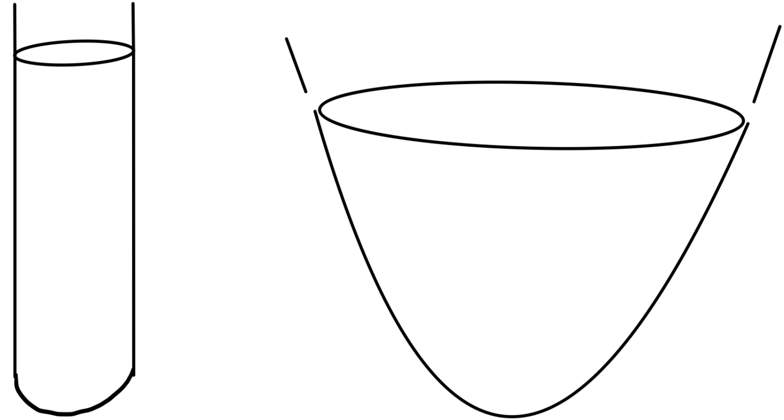
Theorem (Balogh–Kristaly, '22, Math. Ann.)

Let (X, d) be an n-dimensional space with nonnegative curvature. Let us assume (nondegenerate asymptotic cones)

$$\mathrm{AVR}(X, \mathrm{d}) := \lim_{r \to +\infty} \frac{\mathrm{vol}(B_r(x))}{\omega_n r^n} > 0,$$

for some (hence all) $x \in X$. Thus, for every set E of finite perimeter in X, it holds

$$\mathrm{Per}(E) \geq n(\omega_n \mathrm{AVR})^{\frac{1}{n}} \mathrm{vol}(E)^{\frac{n-1}{n}}.$$



AVR = 0

AVR > 0

Existence for large volumes under maximal volume growth

Theorem (A.–Bruè–Fogagnolo–Pozzetta, '22, Calc. Var. PDE)

Let (X, d) be an n-dimensional noncompact nonnegatively curved space, and assume $\mathrm{AVR}(X, \mathrm{d}) > 0$.
Hence isoperimetric sets exist for sufficiently large volumes.

Future directions

Positive mass and foliations of isoperimetric sets

Theorem (Chodosh–Eichmair–Shi–Yu, '21, CPAM)

Let (M, g) be a $C^{2,\frac{1}{2}+\varepsilon}$-asymptotically flat 3-manifold with $R \geq 0$, which is not $\mathbb{R}^3$. Then there is $V_0 > 0$ such that for every $V > V_0$ there exists a unique isoperimetric set with volume V.

$\star$ Linked to the existence of foliations of stable CMC. Related results: [Huisken–Yau, '96, Inv. Math.], [Bray, '97, PhD Thesis], [Eichmair–Metzger, '13, Inv. Math.], [Nerz, '15, Calc. Var.], [Chodosh–Eichmair–Volkmann, '17, JDG], [Yu, '22, Math. Ann.], ...

Positive mass and foliations of isoperimetric sets

Theorem (Chodosh–Eichmair–Shi–Yu, '21, CPAM)

Let (M,g) be a $C^{2,\frac{1}{2}+\varepsilon}$-asymptotically flat 3-manifold with $R \geq 0$, which is not $\mathbb{R}^3$. Then there is $V_0 > 0$ such that for every $V > V_0$ there exists a unique isoperimetric set with volume V.

⋆ Linked to the existence of foliations of stable CMC. Related results: [Huisken–Yau, '96, Inv. Math.], [Bray, '97, PhD Thesis], [Eichmair–Metzger, '13, Inv. Math.], [Nerz, '15, Calc. Var.], [Chodosh–Eichmair–Volkmann, '17, JDG], [Yu, '22, Math. Ann.], ...

⋆ Isoperimetric sets **detect the ADM mass**:

$$\mathfrak{m}_{\mathrm{ADM}} = \lim_{v \to +\infty} \frac{2}{I_M(v)} \left(v - \frac{I_M(v)^{3/2}}{6\sqrt{\pi}} \right).$$

An adaptation to $\mathrm{Ric} \geq 0$, work in progress

Theorem (A.–Bruè–Pozzetta–Semola, Forthcoming)

Let M be a complete n-dimensional Riemannian manifold that does not split such that:

1. $\mathrm{Ric} \geq 0$;
2. *it has a nondegenerate asymptotic cones* $(\mathrm{AVR} > 0)$, *and* $|\mathrm{Riem}| = O(r^{-2})$.

Then there is a set $\mathcal{G} \subset \mathbb{R}^+$ *such that*

$$\frac{\mathcal{L}^1(\mathcal{G} \cap (V, 2V))}{V} \to 1, \text{ as } V \to +\infty,$$

and for every $V \in \mathcal{G}$ *there is a* **unique** *isoperimetric set with volume* V.

Isoperimetric PMT?

Weak notion of $R \geq 0$ for C^0-Riemannian metrics in [Gromov, '14, C. Eur. Math. J.], [Burkhardt–Guim, '19, GAFA], [Huisken, '21, Oberwolfach Report].

Conjecture (Continuous Positive Mass Theorem, Huisken)

Let M be a smooth 3-manifold endowed with a C^0 metric g.

$$R_g \geq 0 \Rightarrow \mathfrak{m}_{\mathrm{iso}} := \sup_{(\Omega_j)\colon P(\Omega_j)\to+\infty} \limsup_{j\to+\infty} \frac{2}{|\partial\Omega_j|}\left(|\Omega_j| - \frac{|\partial\Omega_j|^{3/2}}{6\sqrt{\pi}}\right) \geq 0.$$

Thank you for the attention