Drawstrings and scalar curvature in dimension three

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(joint with Demetre Kazaras)

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Contents

Geroch conjecture and its stability

Theorem (Geroch conjecture)

If g is a smooth metric on T^3 with $R_g \geqslant 0$, then g is flat.

Notation: $T^3 = 3-$torus, $R_g =$ scalar curvature of g.

Proofs:
· Schoen-Yau (minimal surfaces)
· Gromov-Lawson (spinors)
· Stern (harmonic functions)

Geroch conjecture and its stability

Theorem (Geroch conjecture)

If g is a smooth metric on T^3 with $R_g \geqslant 0$, then g is flat.

Problem (stability)

Let g be a metric on T^3 such that

$$R_g \geqslant -\varepsilon, \quad \operatorname{diam}(g) \leqslant D, \quad V_1 \leqslant \operatorname{Vol}(g) \leqslant V_2,$$

where $\varepsilon \ll 1$. What can we conclude about g (is it close to a flat torus in some sense)?

General stability problems

The stability problem can be asked in the context of:

- Positive mass theorem,
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Problem (compactness; Gromov)

Consider the space of manifolds (M, g) *with*

$$R_g \geqslant \lambda, \quad \mathrm{diam}(g) \leqslant D, \quad \mathrm{Vol}(g) \leqslant V, \quad etc...$$

Problems:

· Find a suitable topology under which this space is precompact.

· Find a suitable notion of weak scalar curvature lower bound for the limit spaces.

Stability of Geroch conjecture

Problem

Let g be a metric on T^3 such that

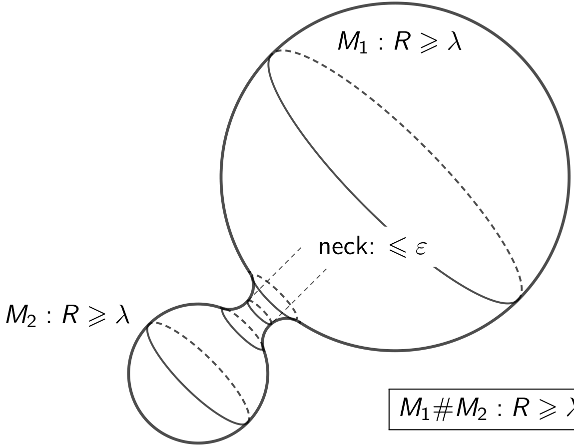
$$R_g \geqslant -\varepsilon, \quad \text{diam}(g) \leqslant D, \quad \text{Vol}(g) \leqslant V,$$

where $\varepsilon \ll 1$. Is g is close to a flat torus?

Answer: extreme examples need to be considered.

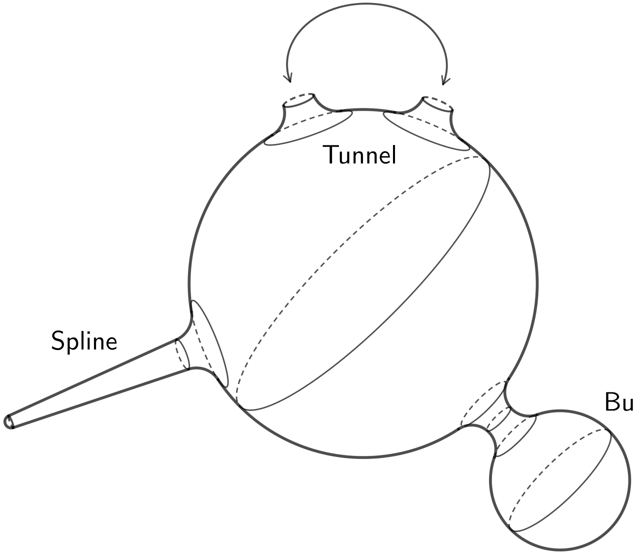
Gromov-Lawson and Schoen-Yau connected sum

See also Basilio-Dodziuk-Sormani, Sweeney Jr. for constructions.



$M_1 \# M_2 : R \geqslant \lambda - \varepsilon$

Induced constructions



Induced constructions

Example 1: Ilmanen's sphere (dense thin splines).
Effect: GH diverging.

Example 2: Basilio-Dodziuk-Sormani sewing (dense small tunnels).
Effect: collapsing a certain subset of the manifold.

minA-IF stability conjecture

$$\text{minA}(M, g) = \inf \left\{ |\Sigma|_g : \Sigma \text{ is a closed minimal surface} \right\}$$

Conjecture (Gromov-Sormani)

Suppose (M_i, g_i) is a sequence of 3-tori, such that

$$\inf_{M_i} R_{g_i} \to 0, \quad \text{minA}(M, g_i) \geqslant A_0 > 0,$$

and

$$\text{diam}(g_i) \leqslant D, \quad \text{Vol}(g_i) \leqslant V.$$

Then a subsequence of M_i converges to a flat T^3 in the (volume-preserving) Sormani-Wenger intrinsic flat sense.

Drawstrings

Theorem (Kazaras–X.)

Let (T^3, g_0) be a fixed flat 3-torus, and γ be a vertical closed geodesic. For any $\varepsilon > 0$ there exists a smooth metric g such that:

(1) $g = g_0$ outside the ε-neighborhood of γ,
(2) $\mathrm{length}_g(\gamma) \leqslant \varepsilon$,
(3) $R_g \geqslant -\varepsilon$.
(4) g has a minA lower bound independent of ε.

We call such construction an ε-*drawstring* around γ.

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Note. Drawstrings can also be constructed in $S^2 \times S^1$ and $H^2 \times S^1$ (with $R \geqslant 2 - \varepsilon$ resp. $R \geqslant -2 - \varepsilon$).

The limit space

Taking $\varepsilon \to 0$, we have:

(1) the length of γ becomes 0.
(2) the remaining part $T^3 \setminus \gamma$ is flat.

The limit space is a *pulled string space*.

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Theorem (Basilio-Dodziuk-Sormani, Basilio-Sormani)

When $\varepsilon \to 0$, the ε-drawstring metrics converge in GH and intrinsic flat sense to a pulled string space

$$X = T^3/(\gamma \sim pt).$$

minA-IF stability conjecture

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Note. Place drawstrings densely $\to$ convergence to a point or zero current. (Difficult to verify minA lower bound.)

The case of dimension $\geqslant 4$

Theorem (Lee-Naber-Neumayer)

Drawstring phenomena exist in dimensions 4 or above.

Namely: given (T^n, g_0) flat torus, $n \geqslant 4$, γ closed geodesic. Then for any $\varepsilon > 0$ we can construct a metric satisfying the same conditions as in our main theorem.

Observations on the codimension

$$\left.\begin{array}{l}\text{Connected sum (and surgery) operation}\\ \text{Lee-Naber-Neumayer construction}\end{array}\right\}$$

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On the other hand, the main theorem occurs in codimension 2.

Related results and questions

Conjecture (disproved by drawstring)

Suppose M (compact 3-manifold) satisfies

$$R \geqslant \lambda > 0, \qquad \text{minA} \geqslant A_0 > 0.$$

Moreover, $\partial M = \Sigma_1 \cup \Sigma_2$, where Σ_i stable minimal spheres. Then there exists $\varepsilon = \varepsilon(R, A_0)$ such that:
(1) if $d(\Sigma_1, \Sigma_2) \leqslant \varepsilon$, then $M \cong S^2 \times S^1$.
(2) if $d(\Sigma_1, \Sigma_2) \leqslant \varepsilon$, then

$$d_H(\Sigma_1, \Sigma_2) \leqslant C(R, A_0) d(\Sigma_1, \Sigma_2).$$

Notation: $d_H =$ Hausdorff distance.

Related results and questions

(M, g) closed hyperbolic 3-manifold. Define the *volume entropy*

$$h(M, g) = \lim_{R \to \infty} R^{-1} \log \left[\mathrm{Vol} \left(\widetilde{B}(x_0, R) \right) \right],$$

where $\widetilde{B}$ denote geodesic balls in the universal cover.

Problem (Agol–Storm–Thurston)

Does $R_g \geqslant -6$ imply $h(M, g) \leqslant 2$?

Theorem (Kazaras–Song–X.)

On any closed hyperbolic 3-manifold M, there exists a metric g with $R_g \geqslant -6$ and $h(M, g) > 2$.

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On any closed hyperbolic 3-manifold M, there exists a metric g with $R_g \geqslant -6$ and $h(M,g) > 2$.

Proof: start with the hyperbolic metric, construct a drawstring around a shortest geodesic in $\pi_1(M)$.

Contents

Part II: construction of drawstrings

Drawstring as a warped product

The drawstring metric is a warped product $g = h + \varphi^2 dt^2$.

Fact: the scalar curvature of such metric is

$$R_g = 2\big(K_h - \frac{\Delta_h \varphi}{\varphi}\big),$$

where $K_h =$ Gauss curvature of h.

Goal: find a metric h and function φ such that
(1) $\{h \text{ is flat}, \varphi = 1\}$ outside a small neighborhood of r_0,
(2) $\varphi(0) \leqslant \varepsilon$,
(3) $\Delta_h \varphi \leqslant (K_h + \varepsilon)\varphi$.

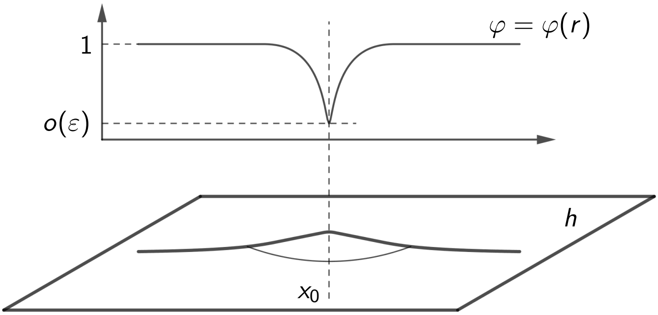
Drawstring in dimension $\geqslant 4$

Lee-Naber-Neumayer construction (h is 3-dimensional):

(1) h forms a *cone* near x_0 (appropriately smoothed),

(2) φ approaches zero near x_0 at the rate of r^δ,

with $\delta \ll \pi - (\text{cone angle}) \ll \varepsilon$.



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In the cone region:

$$R_g = R_h - 2\frac{\Delta_g \varphi}{\varphi} = O(\frac{1}{r^2}) - O(\frac{1}{r^2})$$
$$= +O(\frac{1}{r^2}).$$

Difficulty in dimension 3

Fact: 3-dimensional cone has scalar curvature $O(\frac{1}{r^2})$, while 2-dimensional cone is flat.

$$R_g = 2K_h - 2\frac{\Delta_g \varphi}{\varphi} = 0 - O(\frac{1}{r^2})$$

Smoothing a 2D cone

Observation (smoothing cone heuristic).

The vertex of a 2D cone carries distributional curvature.

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Question: can we smooth the cone to create sufficiently large curvature?

The building block

Lemma (Kazaras-X.)

Consider the metric

$$g = e^{-2u}\left(dr^2 + f(r)^2 d\theta^2\right) + e^{2u} dt^2,$$

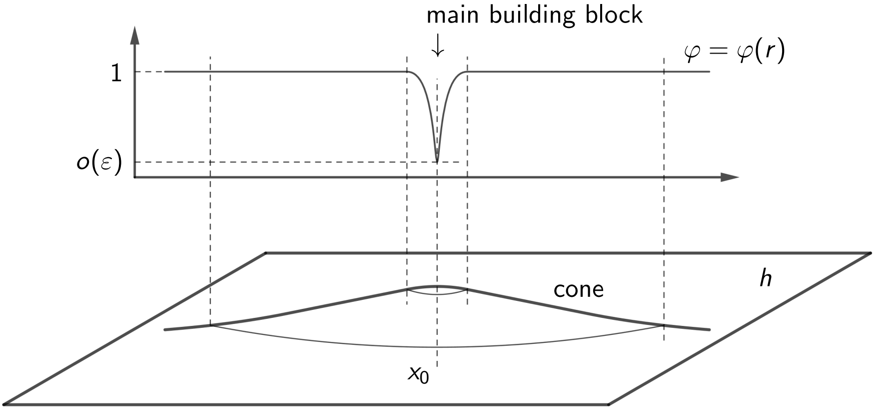
where

$$f(r) = r\left(1 - \frac{c_1}{\log(1/r)}\right), \quad u(r) = -c_2 \log\log(\frac{1}{r})$$

($c_1, c_2 > 0$), then we have

$$R_g = \frac{2}{r^2 \log(1/r)^{2+2c_2}}\left[\frac{c_1(c_1+2)}{\log(1/r) - c_1} + c_1 - c_2^2\right].$$

Global picture of 3D drawstring



Form of the metric?

$$g = e^{-2u}\left(dr^2 + f(r)^2 d\theta^2\right) + e^{2u} dt^2.$$

Fact: the scalar curvature $R_g = 2e^{2u}\left[-\frac{f''}{f} - (u')^2\right]$ does not involve u''.

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This fact (and similar observations) can be tracked in:
· hyperbolic drawstring (Kazaras-Song-X.),
· counterexamples relating intermediate curvature (X.).

Form of the metric?

$$g = e^{-2u}\left(dr^2 + f(r)^2 d\theta^2\right) + e^{2u} dt^2.$$

Facts:

(1) the mean curvature of $\left\{d(-,\gamma) = r\right\}$ is $e^u \frac{f'}{f}$ (containing no derivatives of u).

(2) The expression of R_g does not involve u''.

Higher-dimensional drawstring, collapsing a $(n-2)$-plane in T^n:

$$g = e^{-2(n-2)u}\left(dr^2 + f(r)^2 d\theta^2\right) + e^{2u}\left(dx_1^2 + \cdots + dx_{n-2}^2\right).$$

The warping factors

Question: why the choice

$$f(r) = r\Big(1 - \frac{c_1}{\log(1/r)}\Big), \quad u(r) = -c_2 \log\log(\frac{1}{r})?$$

Conformal inversion heuristic

Ingredient 1

Let $\varphi > 0$ *be a function on* (Σ, g)*, and consider* $\widetilde{g} = \varphi^4 g$*. Then*

$$\Delta_g \varphi \leqslant K_g \varphi \quad \Leftrightarrow \quad \Delta_{\widetilde{g}} \varphi^{-1} \leqslant K_{\widetilde{g}} \varphi^{-1}.$$

Namely: $g + \varphi^2 dt^2$ PSC $\Leftrightarrow$ $\varphi^4 g + \varphi^{-2} dt^2$ PSC.

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Ingredient 2

Let (S^2, g_0) *be the round sphere. There exists function* φ *satisfying* $\Delta\varphi \leqslant \varphi$ *but is arbitrarily large near a point.*

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Example by Sormani-Tian-Wang:

$$\varphi_\delta(r) = \frac{1}{2} \log\left(\frac{1+\delta}{\sin^2 r + \delta}\right) + 1$$

where $\delta \ll 1$.

Combining the ingredients

Consider

$$g = \varphi^4\left(dr^2 + \sin^2 d\theta^2\right) + \varphi^{-2}dt^2, \qquad \varphi(r) = \log\frac{1}{\sin r} + 1.$$

Changing variable $d\widetilde{r} = \varphi dr$, we have

Lemma (Kazaras-X.)

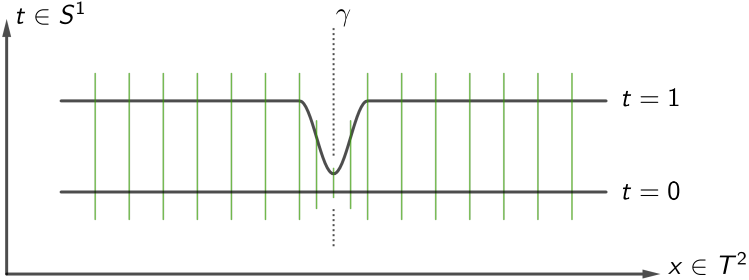
Near $\widetilde{r} = 0$ the leading behavior is

$$g = e^{2\log\log(1/\widetilde{r})+\cdots}\left[d\widetilde{r}^2 + \widetilde{r}^2\left(1 - \frac{1}{\log(1/\widetilde{r})} + \cdots\right)^2 d\theta^2\right]$$
$$e^{-2\log\log(1/\widetilde{r})+\cdots}dt^2.$$

Thank you!

Visualizing drawstrings

The drawstring metric: $g = h + \varphi^2 dt^2$, where $\varphi(x_0) = \varepsilon$ while $\varphi = 1$ outside $B(x_0, \varepsilon)$.



Observations on the codimension

Connected sum (and surgery) operation
Lee-Naber-Neumayer construction

are codimension $\geqslant 3$ phenomena.

Theorem (Gromov-Lawson and Schoen-Yau surgery)

Let $\Sigma^{n-k} \subset M^n$ ($k \geqslant 3$), denote $M' = M \setminus \Sigma$. For any $\varepsilon > 0$ there is a metric g on M' such that:
(1) $g' = g$ outside the ε-neighborhood of Σ,
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(1) $g' = g$ outside the ε-neighborhood of Σ,
(2) $\partial M'$ is a minimal surface.

Related fact: S^{k-1}-bundles ($k \geqslant 3$) admit metrics with $R \gg 1$ (while S^1 bundles may not).

Flat base metric

Supposing h is flat:

$\Rightarrow \Delta_h \varphi \leqslant \varepsilon \varphi$

$\Rightarrow$ Moser's Harnack inequality: $\inf_{B(x_0,1/4)} \varphi \geqslant C \int_{B(x_0,1/2)} \varphi$

$\Rightarrow$ Drawstring does not exist.

Suggest: concentrate large curvature in a region.

The picture of 3D drawstring