The nonlinear potential theory through the looking-glass

and the Penrose Inequality we found there

joint with M. Fogagnolo, L. Mazzieri, A. Pluda and M. Pozzetta

Recent Advances in Comparison Geometry

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MGR in a nutshell 1

NPT and IMCF in comparison 2

Monotonicity formulas 3

The Riemannian Penrose inequalities 4

1

Once upon a time

MGR in a nutshell

- **Einstein (field) equations**: the model of a gravitational system evolving through the time is a Lorentzian manifold $(\mathfrak{M}^{3+1}, \mathfrak{g})$, $\mathfrak{g}$ with signature $(+++-)$, solving the system of equations

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- **Initial data set**: by *[Fourès-Bruhat '52 · ACTA]*, Einstein equations can be interpreted as a system of PDEs for a given initial value (M, g, K), where (M, g) is a Riemannian manifold endowed with a symmetric $(0, 2)$-tensor K satisfying the following constraints

$$\mu = 8\pi T(n, n) = \frac{1}{2}(\mathrm{R} + (\operatorname{tr} K)^2 - |K|^2) \qquad \text{(Energy density)}$$

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- **Dominant energy condition**: generalise the requirement that the energy density is nonnegative $\mu \geq |J| \overset{K=0}{\rightsquigarrow} \mathrm{R} \geq 0$.
- **Time-symmetric**: $K = 0 \rightsquigarrow$ apparent horizons are minimal surfaces.

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Asymptotically flat manifold

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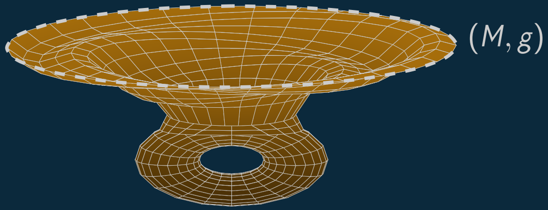

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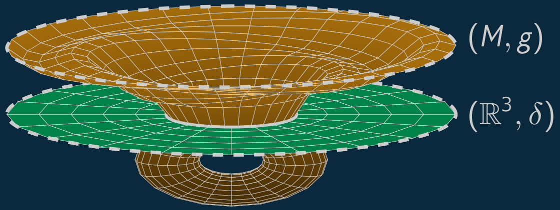

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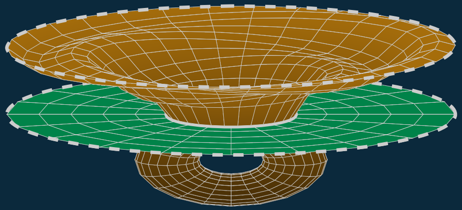
$$|g_{ij} - \delta_{ij}| \leq \mathrm{C}|x|^{-1}$$
$$|\partial g_{ij}| \leq \mathrm{C}|x|^{-2}$$



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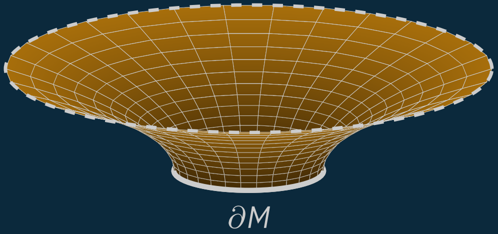
Schwarzschild solution

Given $\mathfrak{m} \geq 0$, the Schwarzschild solution is $(\mathfrak{S}(\mathfrak{m}), \sigma)$, where $\mathfrak{S}(\mathfrak{m}) \cong \mathbb{R}^3 \smallsetminus B_{2\mathfrak{m}}$ and

$$\sigma := \left(1 + \frac{\mathfrak{m}}{2|x|}\right)^4 \delta.$$

Scalar flat ($\mathrm{R} = 0$), asymptotically flat with minimal outermost boundary. The quantity $\mathfrak{m}$ is the mass of the black hole and satisfies

$$\mathfrak{m} = \sqrt{\frac{|\partial M|}{16\pi}}.$$



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- ADM **mass**: defined by *[Arnowitt, Deser, Misner '61]*. *[Bartnik '86]*, *[Chruściel '86]* $\rightsquigarrow$ is a geometric invariant provided (M, g) is $\mathscr{C}^1_\tau$-asymptotically flat, $\tau > 1/2$.

Theorem – [*Schoen, Yau '79 · CMP*]

Let (M, g) a $\mathscr{C}^2_\tau$-asymptotically flat Riemannian manifold, $\tau > 1/2$, with $\mathrm{R} \geq 0$, then $\mathfrak{m}_{\mathrm{ADM}} \geq 0$. Moreover, $\mathfrak{m}_{\mathrm{ADM}} = 0$ if and only if $(M, g) \cong (\mathbb{R}^3, \delta)$.

In dimension $3 \leq n \leq 7$ *[Schoen, Yau '79 · Proc. Nat. Acad. Sci. USA], [Lohkamp '16]*, for spin manifolds *[Witten '81 · CMP], [Bray, Kazaras, Khuri, Stern '22 · J. Geom. Anal]* using harmonic functions with linear growth and *[Agostiniani, Mazzieri, Oronzio '24 · CMP]* using the harmonic Green function.

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RIEMANNIAN PNEROSE INEQUALITY

Theorem – [*Huisken, Ilmanen '01 · JDG*]

Let (M, g) be a $\mathscr{C}^1_1$-asymptotically flat 3-Riemannian manifold with $\mathrm{R} \geq 0$ and Ric $\geq -\mathrm{C}/|x|^2$ *and connected, outermost, minimal boundary. Then*

$$\sqrt{\frac{|\partial M|}{16\pi}} \leq \mathfrak{m}_{\mathrm{ADM}}. \qquad \text{(RPI)}$$

Moreover, the equality holds if and only if $(M, g) \cong (\mathfrak{S}(\mathfrak{m}_{\mathrm{ADM}}), \sigma)$.

For multiple horizons *[Bray '01 · JDG]*, in dimension $3 \leq n \leq 7$ *[Bray, Lee '09 · DMJ]* and *[Agostiniani, Mantegazza, Mazzieri, Oronzio '22]* using nonlinear potential theory.

A “smooth” proof.

Take Σ and evolve it using the IMCF, namely a family of diffeomorphisms $F_t(\Sigma) = \Sigma_t \subset M$ with

$$\frac{\partial}{\partial t} F_t = \frac{\nu}{\mathrm{H}}, \tag{IMCF}$$

where ν is the unit outward pointing vector field and H is the mean curvature of Σ_t.

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Consider the **Hawking mass**

$$\mathfrak{m}_H(\Sigma) := \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \int_\Sigma \frac{\mathrm{H}^2}{16\pi} \, \mathrm{d}\mathcal{H}^2\right). \tag{Hawking mass}$$

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Since $\mathrm{R} \geq 0$, the function $t \mapsto \mathfrak{m}_H(\Sigma_t)$ is monotone nondecreasing, indeed

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathfrak{m}_H(\Sigma_t) = \frac{1}{16\pi}\sqrt{\frac{|\Sigma_t|}{16\pi}} \Bigg(\underbrace{8\pi - \int_{\Sigma_t} \mathrm{R}^\top \,\mathrm{d}\mathcal{H}^2}_{\geq 0 \text{ Gauss-Bonnet}} + \underbrace{\int_{\Sigma_t} |\mathring{\mathrm{h}}|^2 + \mathrm{R} + 2\frac{|\nabla^\top \mathrm{H}|^2}{\mathrm{H}^2} \,\mathrm{d}\mathcal{H}^2}_{Q_1 \geq 0} \Bigg)$$

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Moreover,

$$\mathfrak{m}_H(\Sigma) \leq \varlimsup_{t \to +\infty} \mathfrak{m}_H(\Sigma_t) \underset{\substack{\uparrow \\ \text{asymptotic assumptions on } g}}{\leq} \mathfrak{m}_{\mathrm{ADM}}.$$

$\square$

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Tweedledum and Tweedledee

NPT and IMCF in comparison

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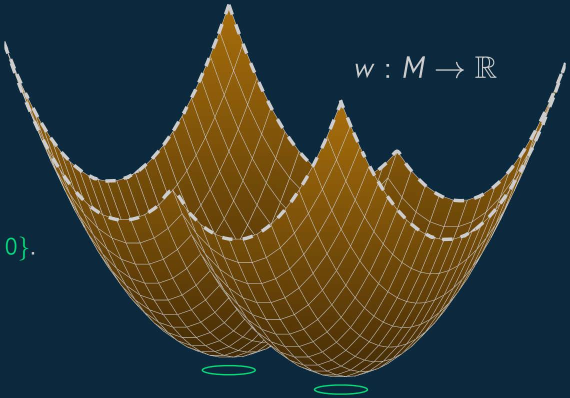

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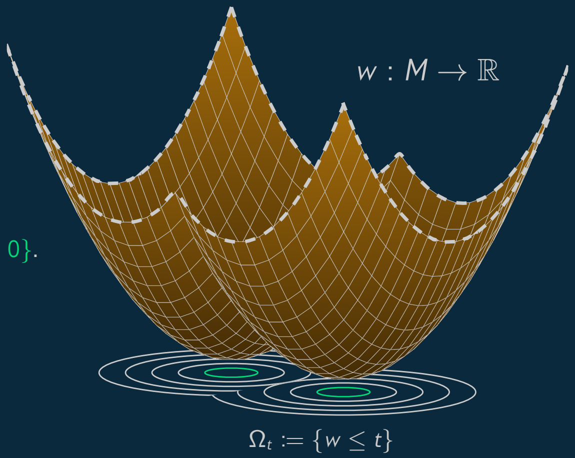

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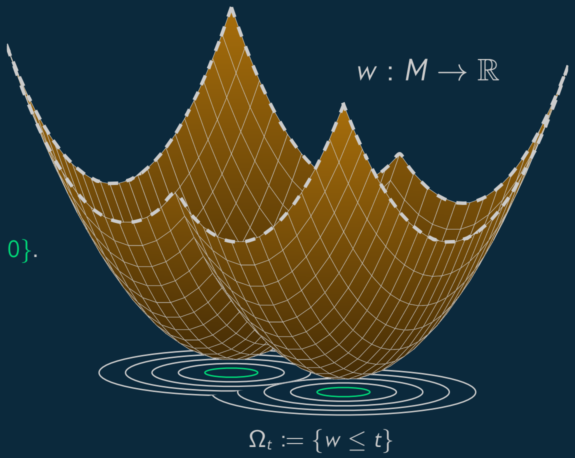

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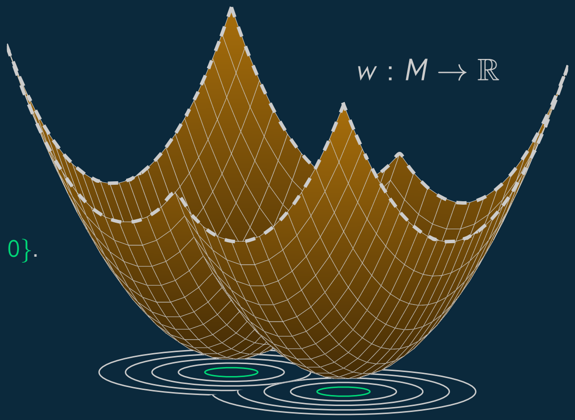

We need to choose the function w wisely

Nonlinear potential theory ($p \leq 2$)

$$\begin{cases} \Delta_p w_p = |\nabla w_p|^p & \text{on } M \smallsetminus \Omega \\ w_p = 0 & \text{on } \partial\Omega \\ w_p \to +\infty & \text{as } |x| \to +\infty \end{cases}$$

where $\Delta_p f = \operatorname{div}(|\nabla w_p|^{p-2} \nabla w_p)$.

Inverse mean curvature flow

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- $\mathscr{C}^{1,\beta}$ everywhere and $\mathscr{C}^\infty$ away from the critical set. Moreover, $|\nabla w_p| \in W^{1,2}$.
- A generic level is almost everywhere regular.
- Defining the (normalised) p-capacity of a set as $\mathfrak{c}_p(K) = \inf\{\mathrm{C}_p \int |\nabla u|^p : u \in \mathscr{C}^\infty_c, u \geq \chi_K\}$,

$$\mathfrak{c}_p(\partial\Omega_t^{(p)}) = \mathrm{e}^t \, \mathfrak{c}_p(\partial\Omega).$$

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- Defining Ω^* the strictly outward minimising hull of Ω,

$$|\partial\Omega_t^{(1)}| = \mathrm{e}^t \, |\partial\Omega^*|.$$

Nonlinear potential theory ($p \leq 2$)

$$\begin{cases} \Delta_p w_p = |\nabla w_p|^p & \text{on } M \smallsetminus \Omega \\ w_p = 0 & \text{on } \partial\Omega \\ w_p \to +\infty & \text{as } |x| \to +\infty \end{cases}$$

where $\Delta_p f = \mathrm{div}(|\nabla w_p|^{p-2} \nabla w_p)$.

- $u_p = e^{-\frac{w_p}{p-1}}$ is p-harmonic in the weak sense.
- $\mathscr{C}^{1,\beta}$ everywhere and $\mathscr{C}^\infty$ away from the critical set. Moreover, $|\nabla w_p| \in W^{1,2}$.
- A generic level is almost everywhere regular.
- Defining the (normalised) p-capacity of a set as $\mathfrak{c}_p(K) = \inf\{ \mathrm{C}_p \int |\nabla u|^p : u \in \mathscr{C}^\infty_c, u \geq \chi_K \}$,

$$\mathfrak{c}_p(\partial\Omega_t^{(p)}) = e^t \, \mathfrak{c}_p(\partial\Omega).$$

What happens sending $p \to 1^+$?

Inverse mean curvature flow

$$\begin{cases} \Delta_1 w_1 = |\nabla w_1| & \text{on } M \smallsetminus \Omega \\ w_1 = 0 & \text{on } \partial\Omega \\ w_1 \to +\infty & \text{as } |x| \to +\infty \end{cases}$$

in this case $\mathrm{H} = |\nabla w_1|$.

- Solution in a nonstandard variational sense *[Huisken, Ilmanen '01 · JDG]*.
- Only Lipschitz.
- A generic level is $\mathscr{C}^{1,1}$ and *strictly outward minimising*.
- Defining Ω^* the strictly outward minimising hull of Ω,

$$|\partial\Omega_t^{(1)}| = e^t \, |\partial\Omega^*|.$$

Proposition – [*Fogagnolo, Mazzieri '22 · JFA*]

In this setting, $\mathfrak{c}_p(\partial\Omega) \to |\partial\Omega^*|/4\pi$ *as* $p \to 1^+$. *In particular,* $\mathfrak{c}_p(\partial\Omega_t^{(p)}) \to |\partial\Omega_t^{(1)}|/4\pi$.

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Theorem – [*Mari, Rigoli, Setti '22 · AJM*]
In this setting, $w_p \to w_1$ *uniformly on compact subsets of* M *as* $p \to 1^+$.

After the works *[Moser '07 · JEMS]* in $\mathbb{R}^n$ and *[Kotschwar, Ni '09 · Ann. Sci. Éc. Norm. Supér]* in nonnegative sectional curvature.

3

The tea party

Monotonicity formulas

[Agostiniani, Mantegazza, Mazzieri, Oronzio '22] introduced the p-**Hawking mass**

$$\mathfrak{m}_H^{(p)}(\Sigma) = \frac{\mathfrak{c}_p(\Sigma)^{\frac{1}{3-p}}}{2}\left[1 + \int_\Sigma \frac{|\nabla w_p|^2}{4(3-p)^2\pi}\,\mathrm{d}\mathcal{H}^2 - \int_\Sigma \frac{|\nabla w_p|\,\mathrm{H}}{4(3-p)\pi}\,\mathrm{d}\mathcal{H}^2\right] \qquad (p\text{-Hawking mass})$$

and proved that $t \mapsto \mathfrak{m}_H^{(p)}(\partial\Omega_t^{(p)})$ is monotone nondecreasing along regular values.

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and proved that $t \mapsto \mathfrak{m}_H^{(p)}(\partial\Omega_t^{(p)})$ is monotone nondecreasing along regular values.

Theorem – **[*B* — , *Pluda, Pozzetta '24*]**

Almost every level of w_p is a curvature varifold and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{m}_H^{(p)}(\partial\Omega_t^{(p)}) \geq \frac{\mathfrak{c}_p(\partial\Omega_t^{(p)})^{\frac{1}{3-p}}}{(3-p)16\pi}\int_{\partial\Omega_t^{(p)}} \underbrace{|\mathring{\mathrm{h}}|^2 + \mathrm{R} + 2\frac{|\nabla^\top|\nabla w_p||^2}{|\nabla w_p|^2} + 2\frac{5-p}{p-1}\left(\frac{|\nabla w_p|}{3-p} - \frac{\mathrm{H}}{2}\right)^2}_{Q_p \geq 0}\,\mathrm{d}\mathcal{H}^2$$

holds for almost every $t \in [0, +\infty)$.

INVERSE MEAN CURVATURE FLOW *[Huisken, Ilmanen '01 · JDG]*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{m}_H(\partial\Omega_t^{(1)}) \geq \frac{1}{16\pi}\sqrt{\frac{|\partial\Omega_t^{(1)}|}{16\pi}} \int_{\partial\Omega_t^{(1)}} |\mathring{\mathrm{h}}|^2 + \mathrm{R} + 2\frac{|\nabla^\top \mathrm{H}|^2}{\mathrm{H}^2} \,\mathrm{d}\mathcal{H}^2$$

NONLINEAR POTENTIAL THEORY *[B — , Pluda, Pozzetta '24]*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{m}_H^{(p)}(\partial\Omega_t^{(p)}) \geq \frac{\mathfrak{c}_p(\partial\Omega_t^{(p)})^{\frac{1}{3-p}}}{(3-p)16\pi} \int_{\partial\Omega_t^{(p)}} |\mathring{\mathrm{h}}|^2 + \mathrm{R} + 2\frac{|\nabla^\top|\nabla w_p||^2}{|\nabla w_p|^2} + 2\frac{5-p}{p-1}\left(\frac{|\nabla w_p|}{3-p} - \frac{\mathrm{H}}{2}\right)^2 \mathrm{d}\mathcal{H}^2$$

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1. $4\pi\,\mathfrak{c}_p(\partial\Omega_t^{(p)}) \to |\partial\Omega_t^{(1)}|$

Inverse mean curvature flow *[Huisken, Ilmanen '01 · JDG]*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{m}_H(\partial\Omega_t^{(1)}) \geq \boxed{\frac{1}{16\pi}\sqrt{\frac{|\partial\Omega_t^{(1)}|}{16\pi}}}\int_{\partial\Omega_t^{(1)}} \boxed{|\mathring{\mathrm{h}}|^2} + \boxed{\mathrm{R}} + \boxed{2\frac{|\nabla^\top \mathrm{H}|^2}{\mathrm{H}^2}}\,\mathrm{d}\mathcal{H}^2$$

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NONLINEAR POTENTIAL THEORY *[B — , Pluda, Pozzetta '24]*

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5. $\int_{\partial\Omega_t^{(p)}} |\nabla w_p| = \int_{\partial\Omega_t^{(p)}} \langle \nabla w_p \,|\, \nu^{(p)}\rangle \overset{1,4}{\sim} \int_{\partial\Omega_t^{(p)}} \langle \vec{\mathrm{H}}^{(p)} \,|\, \nu^{(1)}\rangle \overset{3}{\sim} \int_{\partial\Omega_t^{(1)}} \langle \vec{\mathrm{H}}^{(1)} \,|\, \nu^{(1)}\rangle = \int_{\partial\Omega_t^{(1)}} |\nabla w_1|$

Inverse mean curvature flow *[Huisken, Ilmanen '01 · JDG]*

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6. By coarea: $\lim_{p\to 1^+} \int |\nabla w_p|^2 = \int |\nabla w_1|^2 \rightsquigarrow$ $\nabla w_p \to \nabla w_1$ strongly in L^2

Theorem - [*B —, Pluda, Pozzetta '24*]

In our setting, $\nabla w_p \to \nabla w_1$ *in* L^q_{loc} *for every* $q < +\infty$. *Moreover,* $\partial\Omega_t^{(p)}$ *converges in the sense of varifold to* $\partial\Omega_t^{(1)}$ *and*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{m}_H(\partial\Omega_t^{(1)}) \geq \frac{1}{16\pi}\sqrt{\frac{|\partial\Omega_t^{(1)}|}{16\pi}}\int_{\partial\Omega_t} |\mathring{\mathrm{h}}|^2 + \mathrm{R} + 2\frac{|\nabla^\top \mathrm{H}|^2}{\mathrm{H}^2}\,\mathrm{d}\mathcal{H}^2.$$

for almost every $t \in [0, +\infty)$.

We recover the monotonicity formula proved in *[Huisken, Ilmanen '01 · JDG]*.

4

Fighting the Jabberwocky

Riemannian Penrose inequalities

SCAN & DOWNLOAD

[Huisken '06] introduced the concept of **isoperimetric mass**: given $\{\Omega_k\}$ an exhaustion of M

$$\mathfrak{m}_{\text{iso}} := \sup_{\{\Omega_k\}} \varlimsup_{k\to+\infty} \mathfrak{m}_{\text{iso}}(\Omega_k) \qquad \text{where} \qquad \mathfrak{m}_{\text{iso}}(\Omega_k) := \frac{2}{|\partial\Omega_k|}\underbrace{\left(|\Omega_k| - \frac{|\partial\Omega_k|^{\frac{3}{2}}}{6\sqrt{\pi}}\right)}_{\mathbb{R}^3 \text{ isoperimetric deficit}}.$$

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What about the equivalence with $\mathfrak{m}_{\text{ADM}}$?

Riemannian Penrose inequality is valid for $\mathfrak{m}^{(p)}_{\text{iso}}$ and $\mathfrak{m}_{\text{iso}}$?

Theorem – [*Fan, Shi, Tam '09 · Comm. Anal. Geom.*]
$\mathfrak{m}_{\mathrm{iso}}(B_R) \to \mathfrak{m}_{\mathrm{ADM}}$ as $R \to +\infty$, provided $\mathfrak{m}_{\mathrm{ADM}}$ is defined. In particular, $\mathfrak{m}_{\mathrm{ADM}} \leq \mathfrak{m}_{\mathrm{iso}}$.

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If $\mathfrak{m}_H(\partial\Omega) \leq m$ for Ω in a given class, then $\mathfrak{m}_{\mathrm{iso}} \leq m$.

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Equivalence of masses - RPI
If (M, g) is $\mathscr{C}^1_1$-asymptotically flat and Ric $\geq -\mathrm{C}/|x|^2$

$$\sqrt{\frac{|\partial M|}{16\pi}} \leq \mathfrak{m}_{\text{ADM}} = \mathfrak{m}_{\text{iso}} \leq \mathfrak{m}^{(p)}_{\text{iso}}.$$

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SUMMING UP

Equivalence of masses

always	$\mathfrak{m}_{\text{ADM}} \leq \mathfrak{m}_{\text{iso}}$ $\mathfrak{m}_{\text{ADM}} \leq \mathfrak{m}^{(p)}_{\text{iso}}$
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For all concepts under the ussumptions of *[Huisken, Ilmanen '01 · JDG]*.

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WHAT HAPPENS BELOW THIS THRESHOLD?

Theorem – [*B* — , *Fogagnolo, Mazzieri '22*], [*B* — , *Fogagnolo, Mazzieri '23 · SIGMA*]
Let (M, g) *be a* $\mathscr{C}^1_\tau$*-asymptotically flat* 3*-Riemannian manifold,* $\tau > 1/2$*, with* $\mathrm{R} \geq 0$ *and connected, outermost, minimal boundary. Then,*

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Some remarks

- We already know $\mathfrak{m}_{\mathrm{ADM}} \leq \mathfrak{m}^{(p)}_{\mathrm{iso}}$.

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- It is easy to prove that $\mathfrak{m}^{(p)}_{\mathrm{iso}} \leq \mathfrak{m}_{\mathrm{iso}}$: sharp isoperimetric inequality $\rightsquigarrow$ sharp isocapacitary inequality via symmetrization *[Jauregui '12]* (taking the ball of $\mathbb{R}^3$ of the same volume of $\Omega^{(p)}_t$). Hence, from Shi's isoperimetric inequality

$$|\Omega|^{\frac{3-p}{3}} \leq \left(\frac{4\pi}{3}\right)^{\frac{3-p}{3}} \mathfrak{c}_p(\partial\Omega) + \frac{p(3-p)}{2}\left(\frac{4\pi}{3}\right)^{\frac{3-p}{3}} \mathfrak{c}_p(\partial\Omega)^{\frac{2-p}{3-p}} (\mathfrak{m}_{\mathrm{iso}} + o(1))$$

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as $|\Omega| \to +\infty$.
- If we show $\mathfrak{m}_{\mathrm{iso}} \leq \mathfrak{m}_{\mathrm{ADM}} \rightsquigarrow \mathfrak{m}^{(p)}_{\mathrm{iso}} \leq \mathfrak{m}_{\mathrm{iso}} \leq \mathfrak{m}_{\mathrm{ADM}} \leq \mathfrak{m}^{(p)}_{\mathrm{iso}} \rightsquigarrow$ they are equal.

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- We want to apply *[Jauregui, Lee '19 · CRELLE]*: proving $\mathfrak{m}_H(\partial\Omega) \leq \mathfrak{m}_{\mathrm{ADM}}$ is enough to conclude.

IMCF proof

Take $\Omega \subseteq M$ and evolve with $\Omega_t^{(1)} = \{w_1 \leq t\}$

$$t \mapsto \mathfrak{m}_H(\partial\Omega_t^{(1)})$$

is monotone nondecreasing. By asymptotic assumptions on g

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[Huisken, Ilmanen '01 · JDG]

$$\sqrt{\frac{|\Sigma|}{16\pi}}\left(1 - \int_\Sigma \frac{\mathrm{H}^2}{16\pi}\, \mathrm{d}\mathcal{H}^2\right)$$

Linear potential proof

Take $\Omega \subseteq M$ and evolve with $\Omega_t^{(2)} = \{w_2 \leq t\}$

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[Agostiniani, Mazzieri, Oronzio '24 · CMP]

$$\frac{\mathfrak{c}_2(\Sigma)}{2}\left(1 + \int_\Sigma \frac{(2|\nabla w_2| - \mathrm{H})^2}{16\pi} - \frac{\mathrm{H}^2}{16\pi}\, \mathrm{d}\mathcal{H}^2\right)$$

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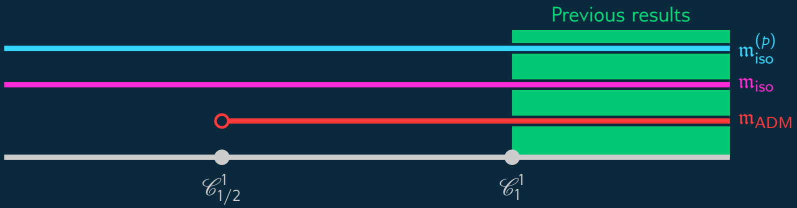
Take Ω, evolve with $\Omega_t^{(1)} = \{w_1 \leq t\}$, at any time t control the Hawking mass with the 2-Hawking mass:

$$\mathfrak{m}_H(\partial\Omega) \leq \varlimsup_{t\to+\infty} \mathfrak{m}_H(\partial\Omega_t^{(1)}) \leq \varlimsup_{t\to+\infty} \frac{\sqrt{|\partial\Omega_t^{(1)}|}}{\sqrt{4\pi}\,\mathfrak{c}_2(\partial\Omega_t^{(1)})}\mathfrak{m}_H^{(2)}(\partial\Omega_t^{(2)}) \leq \varlimsup_{t\to+\infty} \frac{\sqrt{|\partial\Omega_t^{(1)}|}}{\sqrt{4\pi}\,\mathfrak{c}_2(\partial\Omega_t^{(1)})}\mathfrak{m}_{\mathrm{ADM}} \leq \mathfrak{m}_{\mathrm{ADM}}$$


Previous results
$\mathfrak{m}_{\text{iso}}^{(p)}$
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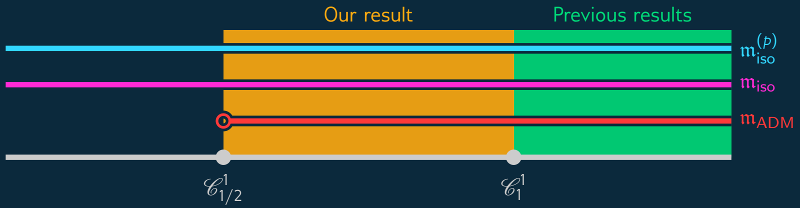
$\mathscr{C}^1_1$


Our result
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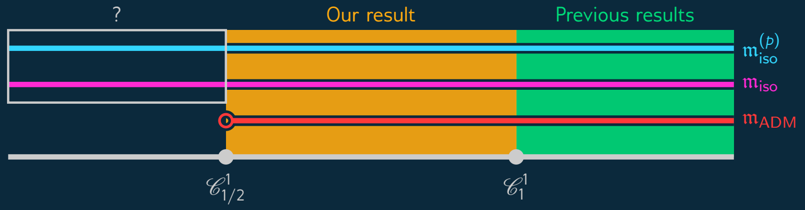
To sum up



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$\mathscr{C}^1_1$

Theorem – [*B —, Fogagnolo, Mazzieri '22*]
Let (M, g) *be a* 3*-Riemannian manifold* $\mathscr{C}^0$*-asymptotically flat with* $\mathrm{R} \geq 0$ *and connected, outermost, minimal boundary. Then,*

$$\sqrt{\frac{|\partial M|}{16\pi}} \leq \mathfrak{m}_{\mathrm{iso}}. \qquad \text{(isoperimetric RPI)}$$

Moreover, the equality holds if and only if $(M, g) \cong (\mathfrak{S}(\mathfrak{m}_{\mathrm{iso}}), \sigma)$.

Proof.

Evolving ∂M using IMCF $\Omega_t = \Omega_t^{(1)} = \{w_1 \leq t\}$ we have

$$\mathfrak{m}_{\mathrm{iso}} \geq \varliminf_{t\to+\infty} \mathfrak{m}_{\mathrm{iso}}(\Omega_t) \geq \varliminf_{t\to+\infty} \frac{2}{|\partial\Omega_t|}\left(|\Omega_t| - \frac{|\partial\Omega_t|^{\frac{3}{2}}}{6\sqrt{\pi}}\right)$$

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$$\overset{\text{de l'Hôpital}}{\geq} \varliminf_{t\to+\infty} \frac{2}{|\partial\Omega_t|}\left(\int_{\partial\Omega_t} \frac{1}{\mathrm{H}}\,\mathrm{d}\mathcal{H}^2 - \frac{|\partial\Omega_t|^{\frac{3}{2}}}{4\sqrt{\pi}}\right)$$

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□

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Let (M, g) be a 3-Riemannian manifold $\mathscr{C}^0$-asymptotically flat with $\mathrm{R} \geq 0$ (+ an extra assumption) and connected, outermost, minimal boundary. Then,

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- With NPT: *[Xia, Yin, Zhou '24 · Adv. Math.]* and *[Mazurowski, Yao '24]* proved a sharp version for the ADM mass $\rightsquigarrow$ wait for Chao Xia's talk.

- The equivalence of masses $\mathfrak{m}_{\text{iso}}^{(p)} = \mathfrak{m}_{\text{iso}} = \mathfrak{m}_{\text{ADM}}$ is proved whenever $\mathfrak{m}_{\text{ADM}}$ is defined. There are cases where $\mathfrak{m}_{\text{ADM}}$ is not defined, but we still have $\mathfrak{m}_{\text{iso}}$ and $\mathfrak{m}_{\text{iso}}^{(p)}$. At this point, we are only able to prove that $\mathfrak{m}_{\text{iso}}^{(p)} \to \mathfrak{m}_{\text{iso}}$ as $p \to 1^+$.

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- These are results towards understanding the geometry of initial data sets endowed with $\mathscr{C}^0$ metrics $\rightsquigarrow$ wait for Gioacchino Antonelli's talk.

Thank you for your attention!