The strong (B)-property for rotation invariant measures

Liran Rotem

Technion – Israel Institute of Technology

Based on joint work with Dario Cordero-Erausquin

Interaction Between Partial Differential Equations and Convex Geometry
"Hangzhou", October 2021

The (B)-property for Gaussian measures

Let γ denote the standard Gaussian measure in $\mathbb{R}^n$,

$$\frac{\mathrm{d}\gamma}{\mathrm{d}x} = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-|x|^2/2}.$$

Let $K \subseteq \mathbb{R}^n$ denote a symmetric convex body (compact convex set $K \subseteq \mathbb{R}^n$ with non-empty interior such that $K = -K$).

Question (Banaszczyk, via Latała)

Is it true that $\gamma\left(\sqrt{ab}K\right)^2 \geq \gamma(aK)\gamma(bK)$ for all $a, b > 0$?

Answer (Cordero–Fradelizi–Maurey, '04)

Yes.

The main goal of today's talk is to extend this result to several non-Gaussian measures.

Log-concavity of measures

Recall that $f : \mathbb{R}^n \to [0, \infty)$ is called log-concave if $(-\log f)$ is a convex function.

Theorem (Prékopa, Leindler, Borell)

If $f : \mathbb{R}^n \to [0, \infty)$ and $\mathrm{d}\mu = f\,\mathrm{d}x$ then μ satisfies the Brunn–Minkowski type inequality

$$\mu\left((1-\lambda)\,A + \lambda B\right) \geq \mu(A)^{1-\lambda}\mu(B)^{\lambda}$$

for all Borel sets $A, B \subseteq \mathbb{R}^n$ and $0 < \lambda < 1$. Here $+$ denotes the Minkowski addition

$$A + B = \{a + b :\ a \in A,\ b \in B\}\,.$$

In particular since $|x|^2/2$ is convex γ is a log-concave measure.

Log-concavity of measures

In particular since $|x|^2/2$ is convex γ is a log-concave measure:

$$\gamma\left((1-\lambda)A+\lambda B\right) \geq \gamma(A)^{1-\lambda}\gamma(B)^{\lambda}.$$

Taking $A = aK$, $B = bK$, $\lambda = \frac{1}{2}$ for a convex body K we obtain

$$\gamma\left(\frac{a+b}{2}K\right)^2 \geq \gamma(aK)\gamma(bK).$$

The theorem of Cordero–Fradelizi–Maurey says that when K is symmetric, $\frac{a+b}{2}$ can be replaced with the smaller $\sqrt{ab}$. In fact they showed more:

Theorem (Cordero–Fradelizi–Maurey, '04)

For every symmetric convex body K the function

$$(t_1, t_2, \ldots, t_n) \mapsto \gamma\left(e^{\Delta(t_1,t_2,\ldots,t_n)}K\right)$$

is log-concave on $\mathbb{R}^n$.

The previous claim follows by restricting to the line $t_1 = t_2 = \cdots = t_n$.

Extensions

It was observed already by C-F-M that the function $t \mapsto \mu(e^t K)$ can sometimes be log-concave when μ is not a Gaussian. For example, we say that K is unconditional if

$$(x_1, x_2, \ldots, x_n) \in K \quad \Longrightarrow \quad (\pm x_1, \pm x_2, \ldots, \pm x_n) \in K,$$

and similarly for measures. If K is an unconditional convex body and μ is an unconditional log-concave measure then
$(t_1, t_2, \ldots, t_n) \mapsto \gamma\left(e^{\Delta(t_1, t_2, \ldots, t_n)} K\right)$ is log-concave.
To avoid repetitions we write:

Definition

- μ has the (B)-property if for every symmetric convex body $K \subseteq \mathbb{R}^n$, $t \mapsto \mu(e^t K)$ is log-concave.
- μ has the strong (B)-property if for every symmetric convex body $K \subseteq \mathbb{R}^n$, $(t_1, t_2, \ldots, t_n) \mapsto \mu\left(e^{\Delta(t_1, t_2, \ldots, t_n)} K\right)$ is log-concave.

What measures have the (strong) (B)-property? Maybe **all** even log-concave measures?

Known Results

- The standard Gaussian measure has the strong (B)-property (C-F-M)
- Certain Gaussian Mixtures have the strong (B)-property (Eskenazis–Nayar–Tkocz '18). In particular $e^{-c|x|^p}\mathrm{d}x$ and $e^{-c\|x\|_p^p}\mathrm{d}x$ have the strong (B)-property **for $\mathbf{0 < p \leq 1}$**. These are not log-concave unless $p = 1$.

The (B)-conjecture is also intimately related to the log-Brunn-Minkowski conjecture:

- If log-BM holds in dimension n then **every** even n-dimensional log-concave measure has the (B)-property (Saroglou '16)
- In particular, every 2-dimensional even log-concave measure has the (B)-property (using Böröczky-Lutwak-Yang-Zhang)
- Conversely, if the uniform measure on $[-1, 1]^n$ has the **strong** (B)-property **for all n**, then log-BM holds (Saroglou '15).

So we have very good reasons to believe that all even log-concave measures have the (B)-property, but very few proven examples.

Some Negative Results

- There exists a convex body $K \subseteq \mathbb{R}^2$ with $0 \in K$ such that $t \mapsto \gamma(e^t K)$ is not log-concave (Nayar–Tkocz '13). So symmetry of K is important.
- There exists an even log-concave measure μ on $\mathbb{R}^2$ which does not have the strong (B)-property (Nayar-Tkocz '19).
- In fact there exist non-standard Gaussian measures with covariance matrix arbitrarily close to Id which don't have the strong (B)-property (Cordero-R., '20).

So we cannot expect all even log-concave measures μ to have the strong (B)-property. It makes sense to impose some symmetry assumptions on μ. Today we will assume μ is rotation invariant.

Our Main Result

Theorem (Cordero-Erausquin, R. '21+)

Let $w : [0, \infty) \to (-\infty, \infty]$ *be an increasing function such that* $t \mapsto w(e^t)$ *is convex. Let* μ *be the measure with density* $\frac{\mathrm{d}\mu}{\mathrm{d}x} = e^{-w(|x|)}$*, and let* $K \subseteq \mathbb{R}^n$ *be a symmetric convex body. Then*

$$(t_1, t_2, \ldots, t_n) \mapsto \mu\left(e^{\Delta(t_1, t_2, \ldots, t_n)} K\right)$$

is log-concave.
In other words, μ *has the strong (B)-property.*

Examples

(Recall: we need $t \mapsto w(e^t)$ to be increasing and convex)

- All rotation invariant log-concave measures have the strong (B)-property.
- In particular, we can take μ to be the uniform measure on the Euclidean ball B_2^n. By applying a linear map we conclude that

$$\left|\sqrt{ab}K \cap \mathcal{E}\right|^2 \geq |aK \cap \mathcal{E}|\,|bK \cap \mathcal{E}|$$

  for all symmetric convex bodies K, all centered ellipsoids $\mathcal{E}$, and all $a, b > 0$.
- One can take $w(t) = c \cdot t^p$ for all $p > 0$ (as $w(e^t) = ce^{pt}$ is convex). Hence all measures $e^{-c\cdot|x|^p}\mathrm{d}x$ have the strong (B)-property. The case $p = 2$ recovers the Gaussian result, and the case $p \leq 1$ recovers the result of Eskenazis–Nayar–Tkocz. Other cases are new.

More Examples

(Recall: we need $t \mapsto w\left(e^t\right)$ to be increasing and convex)

- One can create heavy-tailed distributions with the (B)-property. Taking $w(t) = \beta \cdot \log\left(1 + t^2\right)$ (as $w(e^t) = \beta \log(1 + e^{2t})$ is convex) we conclude the Cauchy-type distribution

$$\mathrm{d}\mu_\beta = \frac{1}{\left(1 + |x|^2\right)^\beta}\mathrm{d}x$$

  has the strong (B)-property.
- By approximation one can create measures with singularities: $\mathrm{d}\mu = \frac{1}{|x|^\beta}\mathrm{d}x$ also has the strong (B)-property as long as $0 < \beta < n$ (to ensure that μ is locally finite).

A corollary

While the roles of μ and K seem different in the theorem, there is in fact some symmetry between them. Instead of assuming μ is rotation invariant, one may assume the same about K:

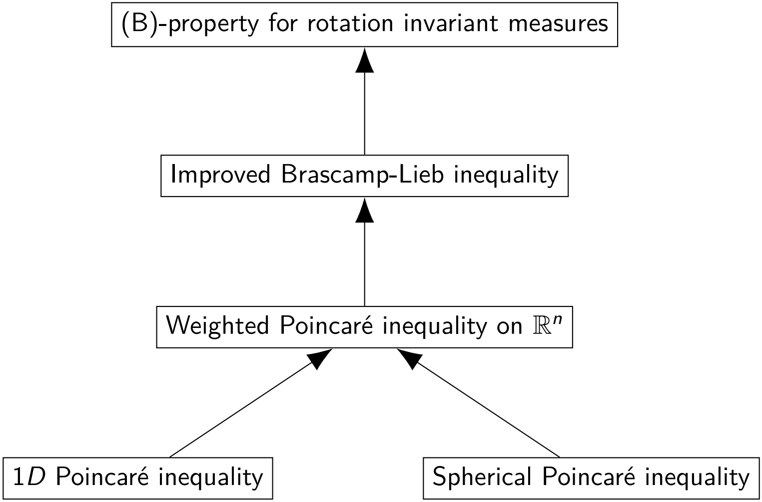
Corollary

Let μ be an even log-concave measure on $\mathbb{R}^n$. Then the function

$$(t_1, t_2, \ldots, t_n) \mapsto \mu\left(e^{\Delta(t_1, t_2, \ldots, t_n)} B_2^n\right)$$

is log-concave.

Proof Sketch



Back to the Gaussian case

How does the proof work in the Gaussian case? We need to show that

$$(t_1, t_2, \ldots, t_n) \mapsto \gamma\left(e^{\Delta(t_1, t_2, \ldots, t_n)} K\right)$$

is log-concave. Restricting to a line, it is enough to show that $\rho(t) = \gamma\left(e^{tA+B} K\right)$ is log-concave for diagonal matrices A and B. Therefore it is enough to show that $(\log \rho)''(t_0) \leq 0$ for all $t_0 \in \mathbb{R}$. By replacing K with $e^{t_0 A+B} K$, we may assume WLOG that $B = 0$ and $t_0 = 0$. Then the condition $(\log \rho)''(0) \leq 0$ becomes

$$\int \langle x, Ax \rangle^2 \, \mathrm{d}\gamma_K - \left(\int \langle x, Ax \rangle \, \mathrm{d}\gamma_K\right)^2 \leq 2 \int |Ax|^2 \, \mathrm{d}\gamma_K.$$

Here γ_K is the Gaussian measure conditioned to belong to K, i.e. $\gamma_K(A) = \frac{\gamma(A \cap K)}{\gamma(K)}$. This is shown by showing that for **every** even function $f : \mathbb{R}^n \to \mathbb{R}$

$$\mathrm{Var}_{\gamma_K} f \leq \frac{1}{2} \int |\nabla f|^2 \, \mathrm{d}\gamma_K.$$

A new Brascamp-Lieb Inequality

When γ is replaced by $\mu = e^{-W(x)}\mathrm{d}x$ one can do the same. The variance inequality one needs to prove is

$$\mathrm{Var}_{\mu_K}\left(\langle \nabla W, Ax\rangle\right) \le \int \left(\left\langle \nabla^2 W \cdot Ax, Ax\right\rangle + \left\langle \nabla W, A^2 x\right\rangle\right) \mathrm{d}\mu_K.$$

What general inequality will imply it?

Theorem

Let $w : [0, \infty) \to \mathbb{R}$ be C^2-smooth and increasing such that $t \mapsto w(e^t)$ is convex. Define $W(x) = w(|x|)$, and let ν be any measure which is even and log-concave with respect to $e^{-W(x)}\mathrm{d}x$. Then for every even function $f : \mathbb{R}^n \to \mathbb{R}$ one has

$$\mathrm{Var}_\nu f \le \int \left\langle \left(\nabla^2 W + \frac{w'(|x|)}{|x|} Id\right)^{-1} \nabla f, \nabla f \right\rangle \mathrm{d}\nu.$$

Remarks

$$\boxed{\mathsf{Var}_\nu f \le \int \left\langle \left(\nabla^2 W + \frac{w'(|x|)}{|x|} Id \right)^{-1} \nabla f, \nabla f \right\rangle \mathrm{d}\nu}$$

- Our assumptions on w imply that $\nabla^2 W + \frac{w'(|x|)}{|x|} Id$ is positive semi-definite.
- In the Gaussian case $w(t) = \frac{1}{2} t^2$ and this inequality becomes $\mathsf{Var}_\gamma f \le \frac{1}{2} \int |\nabla f|^2 \, \mathrm{d}\gamma$ as expected.
- Since $\frac{w'(|x|)}{|x|} Id$ is positive definite this theorem is an improvement of the Brascamp-Lieb inequality

$$\mathsf{Var}_\nu f \le \int \left\langle \left(\nabla^2 W \right)^{-1} \nabla f, \nabla f \right\rangle \mathrm{d}\nu.$$

- In our case $\nabla^2 W$ is a rank-one perturbation of Id so the inverse can be computed explicitly.

Examples

- If $\frac{\mathrm{d}\nu}{\mathrm{d}x} = e^{-|x|^p/p - V(x)}$ for V convex then

$$\mathrm{Var}_\nu f \leq \int \left(\frac{1}{2} |x|^{2-p} |\nabla f|^2 - \frac{p-2}{2p} \cdot \frac{\langle \nabla f, x \rangle^2}{|x|^p} \right) \mathrm{d}\nu$$

for all **even** smooth functions $f : \mathbb{R}^n \to \mathbb{R}^n$. Using the bounds $0 \leq \langle \nabla f, x \rangle^2 \leq |\nabla f|^2 |x|^2$ one deduces

$$\mathrm{Var}_\nu f \leq \max \left\{ \frac{1}{p}, \frac{1}{2} \right\} \cdot \int |x|^{2-p} |\nabla f|^2 \, \mathrm{d}\nu$$

- If $\frac{\mathrm{d}\nu}{\mathrm{d}x} = \frac{1}{\left(1+|x|^2\right)^\beta}$ then

$$\mathrm{Var}_\nu f \leq \frac{1}{4\beta} \int \left(1 + |x|^2\right) \left(|\nabla f|^2 + \langle \nabla f, x \rangle^2 \right) \mathrm{d}\nu$$

From Brascamp-Lieb to weighted Poincaré

Assume in general we want to prove

$$\mathrm{Var}_\mu f \leq \int \left\langle A^{-1}\nabla f, \nabla f \right\rangle \mathrm{d}\mu$$

for $\mathrm{d}\mu = e^{-W(x)}\mathrm{d}x$ and a positive definite A. We assume WLOG that $\int f \mathrm{d}\mu = 0$ and solve $Lu := \Delta u - \nabla W \cdot \nabla u = f$. Integrating by parts our inequality is the same as

$$\int \left\langle \left(A - \nabla^2 W\right) \cdot \nabla u, \nabla u \right\rangle \mathrm{d}\mu \leq \int \left(\left\|\nabla^2 u\right\|_2^2 + \left|A^{-\frac{1}{2}}\nabla f + A^{\frac{1}{2}}\nabla u\right|^2 \right) \mathrm{d}\mu.$$

If $A(x) - \nabla^2 W(x) = c(x) \cdot I$ like in our case then it is enough to prove that

$$\int c \cdot (\partial_i u)^2 \,\mathrm{d}\mu \leq \int |\nabla \partial_i u|^2 \,\mathrm{d}\mu.$$

In our case f and W are even, so u is also even, so every $\partial_i u$ is **odd**.

A new Poincaré inequality

The above discussion explains why the entire result follows from the following:

Theorem

*Let $w : [0,\infty) \to \mathbb{R}$ be C^1-smooth and increasing, and let μ be even and log-concave with respect to $e^{-w(|x|)}\mathrm{d}x$. Then for every **odd** function $h : \mathbb{R}^n \to \mathbb{R}$ one has*

$$\int \frac{w'(|x|)}{|x|} h^2 \mathrm{d}\mu \leq \int |\nabla h|^2 \, \mathrm{d}\mu.$$

In the Gaussian case $w(t) = \frac{1}{2}t^2$ this is the standard Gaussian Poincaré inequality, $\int h^2 \mathrm{d}\gamma \leq \int |\nabla h|^2 \, \mathrm{d}\gamma$, which is well-known.
The main idea of the proof is to integrate in polar coordinates, $x = r\theta$, and combine two Poincaré inequalities - one in r, and one in θ.

The 1-dimensional argument

.

In the r variable, we essentially use the following:

Lemma

Let $f, w : [0, \infty) \to \mathbb{R}$ be smooth functions such that $f(0) = 0$. Then

$$\int_0^\infty \frac{w'}{r} f^2 e^{-w} \mathrm{d}r \leq \int_0^\infty \left(f'\right)^2 e^{-w} \mathrm{d}r.$$

Proof.

Since $f(0) = 0$ we can write $f(r) = rg(r)$ for a smooth function g. Integrating by parts one computes that

$$\int_0^\infty \left(f'\right)^2 e^{-w} \mathrm{d} - \int_0^\infty \frac{w'}{r} f^2 e^{-w} \mathrm{d}r = \int_0^\infty \left(g'\right)^2 r^2 e^{-w} \mathrm{d}r \geq 0.$$

□

The spherical argument

On the unit sphere $\mathbb{S}^{n-1} = \{x : \ |x| = 1\}$ we need the following result:

Proposition

Let $v : \mathbb{R}^n \to \mathbb{R}$ be a convex smooth function and let ν be the measure on $\mathbb{S}^{n-1}$ with density e^{-v}. Then for every smooth $g : \mathbb{S}^{n-1} \to \mathbb{R}$ with $\int_{\mathbb{S}^{n-1}} g \mathrm{d}\nu = 0$ one has

$$\int_{\mathbb{S}^{n-1}} (n - 1 - \langle \nabla v, \theta \rangle) \, g^2 \mathrm{d}\nu \leq \int_{\mathbb{S}^{n-1}} |\nabla_{\mathbb{S}} g|^2 \, \mathrm{d}\nu,$$

where $\nabla_{\mathbb{S}} g$ denotes the spherical gradient.

When $v = 0$ and ν is the Haar measure on $\mathbb{S}^{n-1}$ this reduces to the usual Poincaré inequality on $\mathbb{S}^{n-1}$,

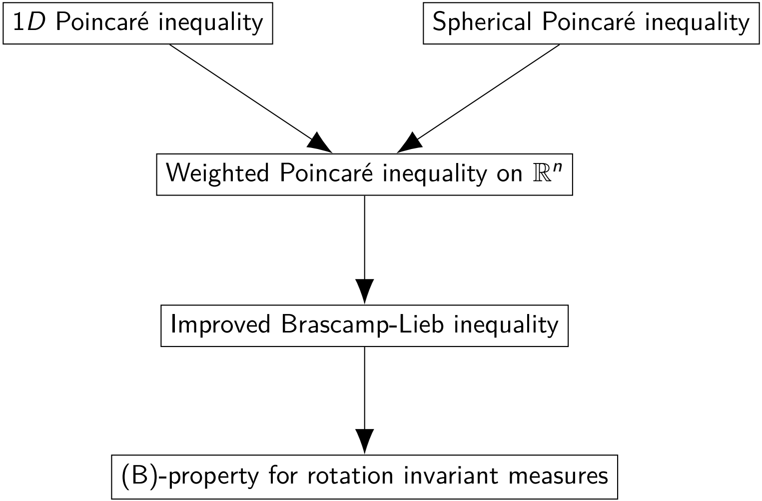
$$\mathrm{Var}_\nu \, g \leq \frac{1}{n-1} \int_{\mathbb{S}^{n-1}} |\nabla_{\mathbb{S}} g|^2 \, \mathrm{d}\nu$$

The spherical argument

$$\boxed{\int_{\mathbb{S}^{n-1}} \left(n-1-\langle \nabla v, \theta\rangle\right) g^2 \mathrm{d}\nu \leq \int_{\mathbb{S}^{n-1}} \left|\nabla_{\mathbb{S}} g\right|^2 \mathrm{d}\nu}$$

- This follows from a general Poincaré inequality of Kolesnikov–Milman on the boundary of weighted Riemannian manifold.
- Their result extends a result of Colesanti. He showed (among other things) that the standard Poincaré inequality on $\mathbb{S}^{n-1}$ is the infinitesimal form of the Brunn–Minkowski inequality.
- In the same way our result is an infinitesimal Prékopa–Leindler inequality: If K_t is the convex body with support function $h_{K_t} = 1 + t \cdot g$, then
$$\rho(t) = \nu\left(K_t\right)$$
is log-concave. Our inequality is exactly the statement $\left(\log \rho\right)''(0) \leq 0$.

Summarizing the argument



The role of symmetry

We only showed that $(t_1, t_2, \dots, t_n) \mapsto \mu\left(e^{\Delta(t_1, t_2, \dots, t_n)} K\right)$ is log-concave for symmetric bodies K. Where did we use the symmetry?

- K is symmetric $\implies$ μ_K is even $\implies$ u from the Brascamp–Lieb proof is even $\implies$ f from the weighted Poincaré is odd.

So the question becomes: Why is it important for the weighted Poincaré that f is odd? Because we integrate in polar coordinates, so we need to know that

$$\int_{\mathbb{S}^{n-1}} f(r\theta) e^{-v(r\theta)} \mathrm{d}\sigma(\theta) = 0$$

for **all** $r > 0$. This is obvious if f is odd and v is even, but difficult to guarantee otherwise.

- It is a natural question if the assumption "f is odd" can be replaced by a weaker assumption that f is "centered" in some sense. It will probably not have any geometric implications.

Thank you!