Affine function valued valuations

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"Interaction Between Partial Differential Equations and Convex Geometry"

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- Why interesting?

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- ◇ Hilbert's third problem in the language of valuations: Is the **volume** (up to scalar multiplications) the only simple and rigid motion invariant valuation on polytopes?
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  - rigid motion invariant: $Z(UK + b) = ZK$, $U \in \mathrm{SO}(n)$, $b \in \mathbb{R}^n$.

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- Applications: Cauchy-Kubota formulas, Crofton formulas, kinematic formulas

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- open for $n \geq 5$ and other spaces

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- From transform of functions, e.g.
  Legendre transform $\mathbf{I}_K \mapsto h_K$, $h_K \mapsto \mathbf{I}_K$
  Laplace transform $\mathbb{1}_K \mapsto \int_K e^{x\cdot y}dy$,
  Fourier transform $\mathbb{1}_K \mapsto \int_K e^{ix\cdot y}dy$

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Groups of affine transforms of $\mathbb{R}^n$

- ◇ special linear group $\mathrm{SL}(n)$
- ◇ general linear group $\mathrm{GL}(n)$

◇ $\mathrm{SL}(n)$ invariant: $Z(\phi K) = ZK$, $K \in \mathcal{K}^n$, $\phi \in \mathrm{SL}(n)$

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Theorem (Blaschke 1937)

$Z : \mathcal{K}^n \to \mathbb{R}$ *is a continuous,* $\mathrm{SL}(n)$ *and translation invariant valuation* $\iff$ *exist* $c_0, c_n \in \mathbb{R}$ *such that*

$$ZK = c_0 V_0(K) + c_n V_n(K)$$

for every $K \in \mathcal{K}_o^n$.

Theorem (Ludwig, Reitzner 2017)

Let $n \geq 2$*. A map* $Z : \mathcal{P}^n \to \mathbb{R}$ *is an* $\mathrm{SL}(n)$ *invariant valuation* $\iff$ *exist* $c_0, c_0', c_0'' \in \mathbb{R}$ *and Cauchy functions* $\xi_1, \xi_2 : [0, \infty) \to \mathbb{R}$

$$ZP = c_0 V_0(P) + c_0'(-1)^{\dim P} V_0(o \cap \operatorname{relint} P) + c_0'' V_0(o \cap P) + \xi_1(V_n(P)) + \xi_2(V_n([P, o]))$$

for every $P \in \mathcal{P}^n$*.*

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- Euler-Schläfli-Poincaré formula

$$V_0(P) = \sum_{j=0}^{\dim P} (-1)^j \sum_{F \in \mathcal{F}_j(P)} V_0(F) = \sum_{j=0}^{\dim P} (-1)^j |\mathcal{F}_j(P)|$$

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- $(-1)^{\dim P} V_0(o \cap \operatorname{relint} P) = ?$ $V_0(o \cap P) = ?$

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Theorem (Li: AM 2021)

Let $n \geq 3$. A map $Z : \mathcal{P}_o^n \to F(\mathbb{R}^n)$ is a continuous and $\mathrm{SL}(n)$ covariant valuation

$\iff$ *exist $\zeta \in C(\mathbb{R})$ and $\mu \in \mathcal{M}^c(\mathbb{R})$ such that*

$$ZP(x) = \begin{cases} \zeta(h_P(x)) + \zeta(-h_{-P}(x)) + \frac{1}{|x|}\int_{\mathbb{R}} V_{n-1}(P \cap H_{x,t})d\mu(t), & x \neq o \\ c_0 V_0(P) + c_n V_n(P), & x = o \end{cases}$$

for every $P \in \mathcal{P}_o^n$ and $x \in \mathbb{R}^n$.

- ◇ continuity of Z: $ZK_i \xrightarrow{p.w.} ZK$ if $K_i \xrightarrow{H} K$
- ◇ $\mathcal{P}_o^n$: polytopes containing the origin
- ◇ $\mathcal{M}^c(\mathbb{R})$: space of signed and continuous Radon measures on $\mathbb{R}$
- ◇ $H_{x,t} = \{y \in \mathbb{R}^n : x \cdot y = t\}$

weighted moment function

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    Ludwig 2005, Wannerer 2011, Haberl 2012, Parapatits 2014

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- Intersection body

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- L_p Busemann-Petty problems:
  Classical $p = -1$: Busemann-Petty, 1956, Lutwak 1988, Ball 1988, Gardner 1994, Zhang 1999
  L_p: Lutwak 1990, Grinberg-Zhang 1999, Yaskin-Yaskina 2006

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◇ $\mathcal{F}_q(\mathbb{R}^n \setminus \{o\})$: $f : \mathbb{R}^n \setminus \{o\} \to \mathbb{R}$ is q-homogeneous if $f(\alpha x) = \alpha^q f(x)$ for any $\alpha > 0$ and $x \in \mathbb{R}^n \setminus \{o\}$.

◇ $t_+ = \max\{t, 0\}$, $t_- = \max\{-t, 0\}$.

Corollary (Ludwig 2005, Haberl 2012, Parapatits 2014)

Let $n \geq 3$ and $q \in \mathbb{R}$. A map $Z : \mathcal{P}_o^n \to \mathcal{F}_q(\mathbb{R}^n \setminus \{o\})$ is a continuous and $\mathrm{SL}(n)$ covariant valuation $\iff$ exist $c_1, c_2, c_3, c_4 \in \mathbb{R}$ s.t.

$$ZP(x) = c_1 h_P(x)^q + c_2 h_{-P}(x)^q + c_3 \int_P (x \cdot y)_+^q dy + c_4 \int_P (x \cdot y)_-^q \, dy,$$

if $q \geq 0$,

$$ZP(x) = c_3 \int_P (x \cdot y)_+^q dy + c_4 \int_P (x \cdot y)_-^q \, dy,$$

if $-1 < q < 0$, and

$$ZP(x) = 0$$

if $q \leq -1$.

Z is q-homogeneous if $Z(\alpha P) = \alpha^q ZP$ for any $\alpha > 0$ and $P \in \mathcal{P}_o^n$.

Corollary

Let $n \geq 3$ and $q \in \mathbb{R}$. A map $Z : \mathcal{P}_o^n \to F(\mathbb{R}^n \setminus \{o\})$ is a continuous, $\mathrm{SL}(n)$ covariant and q-homogeneous valuation
$\iff$ *exist $c_1, c_2, c_3, c_4 \in \mathbb{R}$ s.t.*

$$ZP(x) = c_1 h_P(x)^q + c_2 h_{-P}(x)^q + c_3 \int_P (x \cdot y)_+^{q-n} dy + c_4 \int_P (x \cdot y)_-^{q-n} dy$$

if $q > n-1$,

$$ZP(x) = c_1 h_P(x)^q + c_2 h_{-P}(x)^q$$

if $0 \leq q \leq n-1$, and

$$ZP(x) = 0$$

if $q < 0$.

Without continuity?

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Theorem (Li: AM 2021)

Let $n \geq 3$. A map $Z : \mathcal{P}_o^n \to C_r(\mathbb{R}^n \setminus \{o\})$ is a regular and $\mathrm{SL}(n)$ covariant valuation
$\iff$ *exist $\zeta_1, \zeta_2 \in C(\mathbb{R})$ and $\mu \in \mathcal{M}^c(\mathbb{R})$such that*

$$ZP(x) = \zeta_1(h_P(x)) + \zeta_1^R(h_{-P}(x)) + E_{\zeta_2}^-(P)(x) + E_{\zeta_2^R}^-(-P)(x) + \frac{1}{|x|}\int_{\mathbb{R}} V_{n-1}(P \cap H_{x,t})d\mu(t)$$

for every $P \in \mathcal{P}_o^n$ and $x \in \mathbb{R}^n \setminus \{o\}$.

- $\zeta^R(r) = \zeta(-r)$
- $C_r(\mathbb{R}^n \setminus \{o\})$: the space of continuous functions f on $\mathbb{R}^n \setminus \{o\}$ such that $\lim_{r\to 0} rf(ru) = 0$ for any $u \in S^{n-1}$.
- regularity of Z: $s \mapsto Z(sP)(s^{-1}x)$ is bounded or measurable on any compact interval $I \subset (0, \infty)$

$\diamond$ $\mathrm{GL}_+(n)$ covariant: $Z(\phi K)(x) = ZK(\phi^t x)$, $\phi \in \mathrm{GL}_+(n)$

Theorem (Li: AM 2021)

Let $n \geq 3$. A map $Z : \mathcal{P}_o^n \to F(\mathbb{R}^n)$ is a $\mathrm{GL}_+(n)$ covariant valuation $\Longleftrightarrow$ exist $\zeta_1, \zeta_2 \in F(\mathbb{R})$

$$ZP(x) = \begin{cases} \zeta_1(h_P(x)) + \zeta_1^R(h_{-P}(x)) + E_{\zeta_2}^-(P)(x) + E_{\zeta_2^R}^-(-P)(x), & x \neq o, \\ c_0 V_0(P) + c_0'(-1)^{\dim P} V_0(o \cap \operatorname{relint} P) & x = o \end{cases}$$

for every $P \in \mathcal{P}_o^n$ and $x \in \mathbb{R}^n$.

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for every $P \in \mathcal{P}_o^n$ and $x \in \mathbb{R}^n$.

◇ $\mathbb{1}_{P^\circ}(x) = \zeta(h_P(x))$ for $\zeta(t) = \mathbb{1}_{[0,1]}(t)$

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Euler-type relations

Corollary (Shepherd 1968)

Let $n \geq 0$. The Euler-type relation

$$\sum_{F \in \mathcal{F}(P)} (-1)^{\dim F} \zeta\left(h_F(u)\right) = \zeta(-h_{-P}(x)), \quad x \in \mathbb{R}^n$$

holds for every $P \in \mathcal{P}^n$, $\zeta \in F(\mathbb{R})$ and $x \in \mathbb{R}^n$.

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Corollary

Let $n \geq 1$. We have

$$\sum_{F \in \mathcal{F}(P)} (-1)^{\dim F} \int_{S^{n-1}} \zeta\left(\frac{h_F(u)}{h_L(u)}\right) dV_L(u) = \int_{S^{n-1}} \zeta\left(-\frac{h_{-P}(u)}{h_L(u)}\right) dV_L(u)$$

for every $\zeta \in F(\mathbb{R})$, $P \in \mathcal{P}^n$ and $L \in \mathcal{K}^n$ such that o is contained in the interior of L.

- $\mathcal{F}_j(P) = \{F \in \mathcal{F}(P) : \dim F = j\}$
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Corollary (The local Euler-Schläfli-Poincaré formula)

$$\sum_{j=0}^{\dim P} (-1)^j |\mathcal{F}_j^-(P)| = (-1)^{\dim P} V_0(o \cap \operatorname{relint} P)$$

$$\sum_{j=0}^{\dim P} (-1)^j |\mathcal{F}_j^+(P)| = V_0(o \cap P)$$

$$\sum_{F \in \mathcal{F}(P)} (-1)^{\dim F} V_0(x \cap F) = (-1)^{\dim P} V_0(x \cap \operatorname{relint} P)$$

Valuations on $\mathcal{P}^n$

Theorem (Li: AM 2021)

Let $n \geq 3$. $Z : \mathcal{P}^n \to C_r(\mathbb{R}^n \setminus \{o\})$ *is a regular and* $\mathrm{SL}(n)$ *covariant valuation*
$\iff$ *exist* $\zeta_1, \zeta_2, \zeta_1', \zeta_2' \in C(\mathbb{R})$ *and* $\mu, \mu' \in \mathcal{M}^c(\mathbb{R})$

$$\begin{aligned} &ZP(x) \\ &= E^+_{\zeta_1}(P)(x) + E^+_{\zeta_1^R}(-P)(x) + E^-_{\zeta_2}(P)(x) + E^-_{\zeta_2^R}(-P)(x) \\ &+ E^+_{\zeta_1'}([P,o])(x) + E^+_{(\zeta_1')^R}(-[P,o])(x) + E^-_{\zeta_2'}([P,o])(x) + E^-_{(\zeta_2')^R}(-[P,o])(x) \\ &+ \frac{1}{|x|}\int_{\mathbb{R}} V_{n-1}(P \cap H_{x,t})d\mu(t) + \frac{1}{|x|}\int_{\mathbb{R}} V_{n-1}([P,o] \cap H_{x,t})d\mu'(t) \end{aligned}$$

for every $P \in \mathcal{P}^n$ *and* $x \in \mathbb{R}^n \setminus \{o\}$.

- $\diamond$ $E^-_\zeta(-P) - E^-_\zeta[-P,o] = \zeta(-h_P) - \zeta(-h_{[P,o]})$

Theorem (Li: AM 2021)

Let $n \geq 3$. *A map* $Z : \mathcal{P}^n \to F(\mathbb{R}^n \setminus \{o\})$ *is a continuous and* $\mathrm{SL}(n)$ *covariant valuation*
$\iff$ *exist* $\zeta_1, \zeta_1' \in C(\mathbb{R})$ *and* $\mu, \mu' \in \mathcal{M}^c(\mathbb{R})$

$$ZP(x) = \zeta_1(h_P(x)) + \zeta_1(-h_{-P}(x)) + \frac{1}{|x|}\int_{\mathbb{R}} V_{n-1}(P \cap H_{x,t})d\mu(t)$$
$$+ \zeta_1'(h_{[P,o]}(x)) + \zeta_1'(-h_{-[P,o]}(x)) + \frac{1}{|x|}\int_{\mathbb{R}} V_{n-1}([P,o] \cap H_{x,t})d\mu'(t)$$

for every $P \in \mathcal{P}^n$ *and* $x \in \mathbb{R}^n \setminus \{o\}$.

$\diamond$ $\mathrm{SL}(n)$ contravariant: $Z(\phi K) = ZK \circ \phi^{-1}$, $\phi \in \mathrm{SL}(n)$

Li: IMRN 2020

$Z : \mathcal{P}_o^n \to C(\mathbb{R}^n)$ is a measurable and $\mathrm{SL}(n)$ contravariant valuation

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exist $\zeta \in C(\mathbb{R})$ and $c_0, c_0', c_{n-1} \in \mathbb{R}$ such that

$$Z(P)(x) = \int_{S^{n-1}\setminus N(P,o)} \zeta\left(\frac{x \cdot u}{h_P(u)}\right) dV_P(u) + c_{n-1} V_1(P, [-x, x]) \\ + c_0 V_0(P) + c_0' (-1)^{\dim P} \mathbb{1}_{\mathrm{relint}\, P}(o),$$

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where $\zeta_1(t) = \zeta(t)$ and $\zeta_2(t) = \zeta(-t)$ for every $t \geq 0$.

Corollary (Ludwig 2005, Haberl 2012, Parapatits 2014)

Let $n \geq 3$ *and* $q \geq 0$*. A map* Z*, mapping* $\mathcal{P}_o^n$ *to the space of* q*-homogeneous and continuous functions on* $\mathbb{R}^n$*, is a measurable and* $\mathrm{SL}(n)$ *contravariant valuation*

$\iff$ *exist* $c_{n-1}, \hat{c}_{n-p}^+, \hat{c}_{n-p}^- \in \mathbb{R}$ *s.t.*

$$ZP(x) = c_{n-1}V_1(P, [-x, x]) + \hat{c}_{n-1}^+ \widehat{V}_1(P, [o, x]) + \hat{c}_{n-1}^- \widehat{V}_1(P, [o, -x])$$

if $q = 1$,

$$ZP(x) = \hat{c}_{n-p}^+ \widehat{V}_p(P, [o, x]) + \hat{c}_{n-p}^- \widehat{V}_p(P, [o, -x])$$

if $q > 0$, $q \neq 1$, *and*

$$ZP(x) = c_n V_n(P) + c_0 V_0(P) + c_0'(-1)^{\dim P} \mathbb{1}_{\mathrm{relint}\, P}(o)$$

if $q = 0$.

Characterization of L_p mixed volume

- $\diamond$ $Z : \mathcal{P}_o^n \times \mathcal{K}_c^n \to \mathbb{R}$ is $\mathrm{SL}(n)$ invariant
  $$\Longleftrightarrow Z(\phi P, \phi L) = Z(P, L), \quad \phi \in \mathrm{SL}(n),\ P \in \mathcal{P}_o^n,\ L \in \mathcal{K}_c^n$$
- $\diamond$ L_p additive with respect to the second variable:
  $Z(P, L_1 +_p L_2) = Z(P, L_1) + Z(P, L_2),\ \ P \in \mathcal{P}_o^n, L_1, L_2 \in \mathcal{K}_c^n.$

Corollary (Li: IMRN 2020)

Let $p \geq 1$ and p not an even integer. $Z : \mathcal{P}_o^n \times \mathcal{K}_c^n \to \mathbb{R}$ is an $\mathrm{SL}(n)$ invariant map which is a measurable valuation with respect to the first variable and continuous and L_p additive with respect to the second variable

$$\Longleftrightarrow$$

exist $\widehat{c}_{n-p}, c_{n-1} \in \mathbb{R}$ such that

$$Z(P, L) = \widehat{c}_{n-p} \widehat{V}_p(P, L) + c_{n-1} \delta_p^1 V_1(P, L)$$

for every $P \in \mathcal{P}_o^n$ and $L \in \mathcal{K}_c^n$.

Characterization of L_p mixed volume

- $Z: \mathcal{P}_o^n \times \mathcal{K}_c^n \to \mathbb{R}$ is $\mathrm{SL}(n)$ invariant
$$\Longleftrightarrow Z(\phi P, \phi L) = Z(P, L), \quad \phi \in \mathrm{SL}(n),\ P \in \mathcal{P}_o^n,\ L \in \mathcal{K}_c^n$$
- L_p additive with respect to the second variable:
$Z(P, L_1 +_p L_2) = Z(P, L_1) + Z(P, L_2), \ \ P \in \mathcal{P}_o^n, L_1, L_2 \in \mathcal{K}_c^n.$

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Questions: characterizations of L_p mixed volumes and Orlicz mixed volumes

Sketch of Proofs (the covariant case)

- ◇ reduce to simplices

Lemma

Let Z and Z' be $\mathrm{SL}(n)$ covariant function valued valuations on $\mathcal{P}_o^n$. If $Z(s[o, e_1, \dots, e_d]) = Z'(s[o, e_1, \dots, e_d])$ for every $s > 0$ and $0 \le d \le n$, then $ZP = Z'P$ for every $P \in \mathcal{P}_o^n$.

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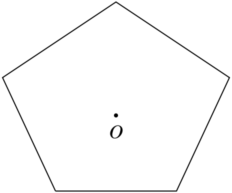

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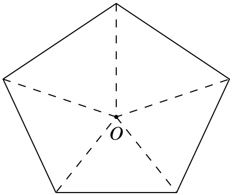

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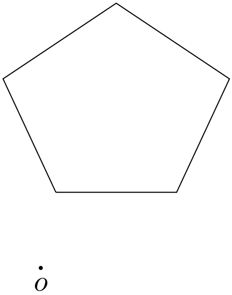

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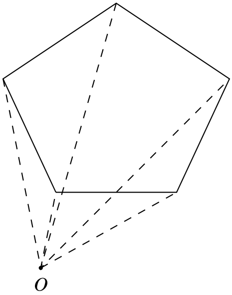
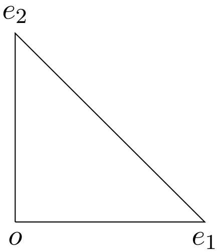


$d \geq 2$

$T^d = [o, e_1, \ldots, e_d]$



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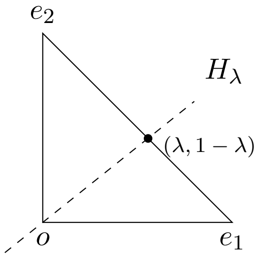

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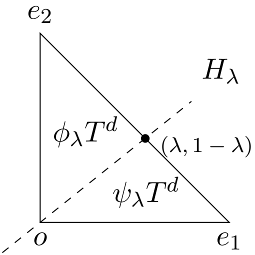

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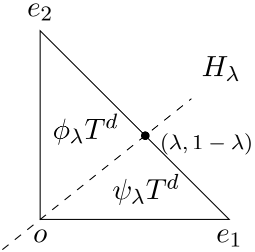

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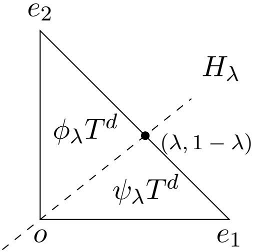

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$$\phi_\lambda e_i = e_i, \quad 3 \leq i \leq n,$$

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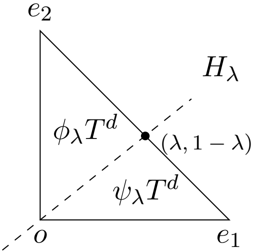

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◇ Vanishes on $(d-1)$-dimension + $\mathrm{SL}(n)$ covariance $\Longrightarrow$

$$Z(s^{1/n}T^d)(x) = Z\big((\lambda s)^{1/n}T^d\big)\big(\lambda^{-1/n}\phi_\lambda^t x\big) + Z\big(((1-\lambda)s)^{1/n}T^d\big)\big(((1-\lambda)s)^{-1/n}\psi_\lambda^t x\big)$$

Uniqueness

Lemma

Let $n \geq 3$ *and* $Z : \mathcal{P}_o^n \to C(\mathbb{R}^n \setminus \{o\})$ *be a regular* $\mathrm{SL}(n)$ *covariant valuation. If*

$$Z\{o\}(e_n) = ZT^1(re_1) = ZT^2(re_1) = ZT^{n-1}(re_n) = Z(sT^n)(re_1) = 0$$

for every $r \neq 0$ *and* $s > 0$*, then* $Z = 0$.

Lemma

Let $n \geq 3$. *If* $Z : \mathcal{P}_o^n \to C_r(\mathbb{R}^n \setminus \{o\})$ *is a regular, simple and* $\mathrm{SL}(n)$ *covariant valuation, then there is a measure* $\mu \in \mathcal{M}^c(\mathbb{R})$ *such that*

$$Z(sT^n)(re_1) = \frac{1}{|r|} \int_{\mathbb{R}} V_{n-1}(sT^n \cap H_{re_1,t}) d\mu(t)$$

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Proofs:

1. $Z(sT^n)(re_1) = s^n Z(T^n)(rse_1)$
2. $g(r) := r^n Z(T^n)(re_1) = \frac{1}{(n-1)!} \int_0^r (r-t)^{n-1} d\mu(t),\ r \neq 0$
   $\Longleftrightarrow$ $(*)$ $g^{(k)}$ exists and is continuous on $\mathbb{R} \setminus \{o\}$ from $k = 1$ up to $n - 1$ and

   $$\lim_{r \to 0} g^{(k)}(r) = 0$$

   for $0 \leq k \leq n - 1$.

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3. Show that $(*)$ holds.

Thank you!