Curves in affine and Semi-Euclidean spaces

Huili Liu (刘会立)
Department of Mathematics,
Northeastern University

Interaction Between Partial Differential Equations and Convex Geometry (Online)

IASM-BIRS, October 21, 2021,

Purpose

- Curves in Semi-Euclidean spaces.
- Curves in affine space.
- Some special curves.
- Relations of curves.

Preliminaries

(1) Summary of

curves, spherical curves

in Euclidean 3-space and

in Minkowski 3-space.

Preliminaries-Curves

Let $x(s) : \mathbf{I} \to \mathbb{E}^3$ be a regular space curve in Euclidean 3-space $\mathbb{E}^3$ parameterized by its arc length s. Denote by $\{\alpha(s), \beta(s), \gamma(s)\}$ the Frenet frame field along $x(s)$, that is, $\alpha(s)$ is the tangent vector field, $\beta(s)$ the normal vector field and $\gamma(s)$ the binormal vector field of $x(s)$, respectively.

The Frenet formulas are given by

$$\begin{cases} \dfrac{\mathrm{d}\alpha(s)}{\mathrm{d}s} = \dot{\alpha}(s) = \kappa(s)\beta(s), \\ \dfrac{\mathrm{d}\beta(s)}{\mathrm{d}s} = \dot{\beta}(s) = -\kappa(s)\alpha(s) + \tau(s)\gamma(s), \\ \dfrac{\mathrm{d}\gamma(s)}{\mathrm{d}s} = \dot{\gamma}(s) = -\tau(s)\beta(s). \end{cases} \tag{1}$$

Where $\kappa(s)$ and $\tau(s)$ are the curvature function and torsion function of the curve $x(s)$ in $\mathbb{E}^3$.

If $x(s)$ is a spherical curve, by a translation in $\mathbb{E}^3$ if necessary, we may assume that $\langle x(s), x(s) \rangle = \langle x, x \rangle = a^2$. Here $a > 0$ is constant and $\langle \quad . \quad \rangle$ denotes the standard inner product of $\mathbb{E}^3$. Without loss of generality we may assume that $a = 1$.

Let $\alpha(s) := \dot{x}(s)$ and $y(s) := \alpha(s) \times x(s)$, here $\times$ denotes the vector product of two vectors in $\mathbb{E}^3$. Then $\alpha(s)$, $x(s)$ and $y(s)$ form an orthonormal basis along the curve $x(s)$ in $\mathbb{E}^3$. We call $\{\alpha(s), x(s), y(s)\}$ the spherical Frenet frame of spherical curve $x(s)$ in $\mathbb{E}^3$.

Then there exists a function $\kappa_g(s)$ such that

$$\begin{cases} \dfrac{\mathrm{d}\alpha(s)}{\mathrm{d}s} = \dot{\alpha}(s) = -x(s) + \kappa_g(s)y(s), \\ \dfrac{\mathrm{d}x(s)}{\mathrm{d}s} = \dot{x}(s) = \alpha(s), \\ \dfrac{\mathrm{d}y(s)}{\mathrm{d}s} = \dot{y}(s) = -\kappa_g(s)\alpha(s). \end{cases} \tag{2}$$

We call $\kappa_g(s)$ spherical curvature function of the spherical curve $x(s)$ in $\mathbb{E}^3$. For the spherical curve $x(s)$, the curvature function $\kappa(s)$, torsion function $\tau(s)$ and spherical curvature function $\kappa_g(s)$ satisfy

$$\begin{cases} \kappa(s) = \sqrt{1+\kappa_g^2}, \\ \tau(s) = \dfrac{\pm\kappa_g'}{1+\kappa_g^2}. \end{cases} \tag{3}$$

Let $x : \mathbf{I} \to \mathbb{E}^3$ be a regular space curve in Euclidean 3-space $\mathbb{E}^3$. If the position vector of the curve x always lies in its rectifying plane, the curve x is called a rectifying curve.

The curvature function $\kappa(s)$ and torsion function $\tau(s)$ of the rectifying curve $x(s)$ with the arc length parameter s satisfy

$$\tau(s)/k(s) = as + b$$

for some constants $a \neq 0$ and b.

The position vector field $x(s)$ of the rectifying curve can be written as

$$x(t) = (a \sec t) x_0(t),$$

where $x_0(t)$ is a spherical curve with the arc length parameter t in $\mathbb{E}^3$, a is constant ([C]).

Let $x : \mathbf{I} \to \mathbb{Q}^2 \subset \mathbb{E}^3_1$ be a regular spacelike curve in a two dimensional lightlike cone $\mathbb{Q}^2$ of the Minkowski 3-space $\mathbb{E}^3_1$ with arc length parameter s. Putting ($\ddot{x}(s) = \frac{\mathrm{d}^2 x(s)}{\mathrm{d}s^2}$)

$$y(s) = -\ddot{x}(s) - \frac{1}{2}\langle \ddot{x}(s), \ddot{x}(s)\rangle x(s), \quad (4)$$

we have $\langle x, x\rangle = \langle y, y\rangle = 0$, $\langle x, y\rangle = 1$.

Using $\alpha(s) = \dot{x}(s)$ we know that $\{x(s), \alpha(s), y(s)\}$ forms an asymptotic orthonormal frame along the curve $x(s)$ in $\mathbb{E}^3_1$, and the cone Frenet formulas of $x(s)$ are given by ([LIU-Curves])

$$\begin{cases} \dot{x}(s) = \alpha(s), \\ \dot{\alpha}(s) = \kappa_g(s)x(s) - y(s), \\ \dot{y}(s) = -\kappa_g(s)\alpha(s). \end{cases} \tag{5}$$

Cone curves

We call $\kappa_g(s)$ cone curvature function of the cone curve $x(s)$ in $\mathbb{Q}^2 \subset \mathbb{E}^3_1$.

Remark

We use the same function κ_g to denote the spherical curvature function of the spherical curve in Euclidean space and the cone curvature function of the cone curve in Minkowski space.

We need the following lemma of the solutions for the second order linear differential equations (see [P-Z], 0.2.1-1, p21).

Lemma 2

Let $f_1(s)$ be any nontrivial partial solution of the equation

$$\lambda_2(s)f'' + \lambda_1(s)f' + \lambda_0(s)f = 0, \tag{6}$$

where, $f'' = f''(s) = \frac{\mathrm{d}^2 f(s)}{\mathrm{d}s^2}$, $f' = f'(s) = \frac{\mathrm{d}f(s)}{\mathrm{d}s}$. *Then*

$$f_2(s) = f_1 \int \frac{e^{-\Lambda}}{f_1^2} \mathrm{d}s, \quad \Lambda = \int \frac{\lambda_1}{\lambda_2} \mathrm{d}s \qquad (7)$$

is also the nontrivial solution of (6) *and* $f_1(s)$ *and* $f_2(s)$ *are a fundamental system of solutions of* (6). *That is, the solutions of* (6) *can be written as*

$$f(s) = c_1 f_1(s) + c_2 f_2(s),$$

where c_1 *and* c_2 *are arbitrary constants.*

(2) Theories of

Centroaffine curves

in affine *n*-space.

Centroaffine curves

In this section we define centroaffine arc length parameter and curvature functions of a curve immersion in an affine space.

Let $x = x(t) : \mathbf{I} \to \mathbb{A}^{n+1}$ be a curve immersion in an affine $(n+1)$-space $\mathbb{A}^{n+1}$ with arbitrary parameter t, where $\mathbf{I}$ is a real interval and $[\ \ , \ldots , \ \ ]$ the standard determinant in $\mathbb{A}^{n+1}$. Without loss of generality we may assume that 0 lies in $\mathbf{I}$.

For a parameter transformation of $x(t) = x(\sigma)$, $\sigma = \sigma(t)$, by a direct calculation we have

$$
\begin{aligned}
x' = \frac{\mathrm{d}x}{\mathrm{d}t} &= \dot{x}\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}t}, \\
x'' &= \ddot{x}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 + \dot{x}\frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2}, \\
&\cdots\cdots \\
x^{(k)} &= \overset{(k)}{x}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^k + \ \mathrm{mod}\ (\overset{(k-1)}{x}, \overset{(k-2)}{x}, \ldots, \dot{x}).
\end{aligned}
$$

Then

$$[x', x'', \ldots, x^{(n)}, x^{(n+1)}] = [\dot{x}, \ddot{x}, \ldots, \overset{(n)}{x}, \overset{(n+1)}{x}] \left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^{\frac{(n+1)(n+2)}{2}}, \quad (8)$$

$$\left[x', x'', \ldots, x^{(n)}, x\right] = [\dot{x}, \ddot{x}, \ldots, \overset{(n)}{x}, x] \left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^{\frac{n(n+1)}{2}}. \quad (9)$$

We always assume that the curve x is centroaffine regular, that is

$$[x', x'', \ldots, x^{(n)}, x] \neq 0.$$

And the condition

$$[x', x'', \ldots, x^{(n)}, x^{n+1}] \neq 0$$

means that x is proper or non degenerated in $\mathbb{A}^{n+1}$.

From (8) and (9) we get

$$\left(\frac{[x', x'', \ldots, x^{(n)}, x^{(n+1)}]}{[x', x'', \ldots, x^{(n)}, x]}\right) \mathrm{d}t^{n+1} \qquad (10)$$

$$= \left(\frac{[\dot{x}, \ddot{x}, \ldots, \overset{(n)}{x}, \overset{(n+1)}{x}]}{[\dot{x}, \ddot{x}, \ldots, \overset{(n)}{x}, x]}\right) \mathrm{d}\sigma^{n+1}.$$

Therefore by (10) we know that

$$
\begin{aligned}
\mathrm{d}s &= e^{\sigma}\mathrm{d}t \\
&= \left| \frac{[x', x'', \ldots, x^{(n)}, x^{(n+1)}]}{[x', x'', \ldots, x^{(n)}, x]} \right|^{\frac{1}{n+1}} \mathrm{d}t
\end{aligned} \tag{11}
$$

is global defined (no dependent on the choose of the parameter) and invariant under centroaffine transformations.

Remark

Putting $\sigma = -t$ we know that (10) is also true. But the signs of $\mathrm{d}t^{n+1}$ and

$$\frac{[x'(t), x''(t), \ldots, x^{(n)}(t), x^{(n+1)}(t)]}{[x'(t), x''(t), \ldots, x^{(n)}(t), x(t)]}$$

can be the same and also can be different. Therefore the right side of the equation (11) is depended on the sign of the parameter t.

Definition 1

Let $x = x(s) : \mathbf{I} \to \mathbb{A}^{n+1}$ *be a curve immersion in an affine* $(n+1)$*-space* $\mathbb{A}^{n+1}$ *with a parameter* s*. If*

$$\frac{[x', x'', \ldots, x^{(n)}, x^{(n+1)}]}{[x', x'', \ldots, x^{(n)}, x]} = \varepsilon = \pm 1, \qquad (12)$$

the parameter s *is called* ***centroaffine arc length parameter*** *of the curve* $x(s)$*.*

For the convenience, putting $e_1 = x'$, $e_2 = x''$, $\ldots$, $e_n = x^{(n)}$, we have

$$x^{(n+1)} = \kappa_1 e_1 + \kappa_2 e_2 + \cdots + \kappa_n e_n + \varepsilon x, \tag{13}$$

$\varepsilon = \pm 1$ and for $1 \leq i \leq n$,

$$\kappa_i = \frac{[e_1, \ldots, e_{i-1}, x^{(n+1)}, e_{i+1}, \ldots, e_n, x]}{[e_1, \ldots, e_{i-1}, e_i, e_{i+1}, \ldots, e_n, x]}. \tag{14}$$

Definition 2

Let $x = x(s) : \mathbf{I} \to \mathbb{A}^{n+1}$ *be a curve immersion in an affine* $(n+1)$*-space* $\mathbb{A}^{n+1}$ *with the centroaffine arc length parameter* s*. The function* $\kappa_i(s)$ *defined by* (14) *is called* ***i-th centroaffine curvature function*** *of the curve* $x(s)$*.*

Remark

The functions $\kappa_i(s)$, $i \geq 2$ can also be called $(i-1)$-th centroaffine torsion functions of the curve $x(s)$.

Remark

From (14) we know that

$$\kappa_i(-s) = (-1)^{n+1-i}\kappa_i(s). \tag{15}$$

Remark

If $\kappa_n \equiv 0$, *s is also equiaffine arc length* ([L-S-Z], [S-S-V]) *since*

$$[e_1, e_2, \ldots, e_n, e_{n+1}] = \varepsilon[e_1, e_2, \ldots, e_n, x] = \pm 1$$

gives

$$[e_1, e_2, \ldots, e_{n-1}, e_{n+1}, x] = 0.$$

(3) **Centroaffine curves**

in affine 2-space.

At first we consider the centroaffine curves in affine 2-space $\mathbb{A}^2$. For the centroaffine curve $x = x(s) : \mathbf{I} \to \mathbb{A}^2$, from (14) we have

$$\kappa_1(s) = \frac{[e_2, x]}{[e_1, x]} = \frac{[x'', x]}{[x', x]}. \qquad (16)$$

Then (13) becomes

$$x'' - \kappa_1 x' - \varepsilon x = 0, \quad \varepsilon = \pm 1. \qquad (17)$$

Therefore we have the following

Theorem 7

(1) *If* $\kappa_1 = a =$ *constant*, *solving* $x'' - ax' - \varepsilon x = 0$ *we get*
(*a*) $x(s) = C_1 \exp\left(\frac{a+\lambda}{2}s\right) + C_2 \exp\left(\frac{a-\lambda}{2}s\right)$, *for* $\lambda^2 = a^2 + 4\varepsilon > 0$;
(*b*) $x(s) = \exp\left(\frac{1}{2}as\right)\left(C_1 \cos\left(\frac{1}{2}\lambda s\right) + C_2 \sin\left(\frac{1}{2}\lambda s\right)\right)$, *for* $-\lambda^2 = a^2 + 4\varepsilon < 0$, $C_1, C_2 \in \mathbb{A}^2$;
(*c*) $x(s) = \exp\left(\frac{1}{2}as\right)(C_1 s + C_2)$, *for* $a^2 = -4\varepsilon$, $C_1, C_2 \in \mathbb{A}^2$.
(2) *If* $\kappa_1 = -s$, *solving* $x'' + sx' - \varepsilon x = 0$ *we have*
(*a*) $x(s) = \exp\left(-\frac{1}{2}s^2\right)\left(C_1 + C_2 \int \exp\left(\frac{1}{2}s^2\right) \mathrm{d}s\right)$, *for* $\varepsilon = -1$, $C_1, C_2 \in \mathbb{A}^2$;
(*b*) $x(s) = C_1 s + C_2\left[\exp\left(-\frac{1}{2}s^2\right) + s\int \exp\left(-\frac{1}{2}s^2\right) \mathrm{d}s\right]$, *for* $\varepsilon = 1$, $C_1, C_2 \in \mathbb{A}^2$.
(3) *If* $\kappa_1 = s$, *solving* $x'' - sx' - \varepsilon x = 0$ *we have*
(*a*) $x(s) = C_1 s + C_2[-\exp\left(\frac{1}{2}s^2\right) + s\int \exp\left(\frac{1}{2}s^2\right) \mathrm{d}s]$, *for* $\varepsilon = -1$, $C_1, C_2 \in \mathbb{A}^2$;
(*b*) $x(s) = \exp\left(\frac{1}{2}s^2\right)\left(C_1 + C_2 \int \exp\left(-\frac{1}{2}s^2\right)\right)$, *for* $\varepsilon = 1$, $C_1, C_2 \in \mathbb{A}^2$.

Proof.

(1) $x'' - ax' - \varepsilon x = 0$ is a linear differential equation of second order with constant coefficients.

(2a) For $x'' + sx' + x = (x' + sx)' = 0$, then $x' + sx = 0$ gives $x = C_1 \exp\left(-\frac{1}{2}s^2\right)$.

(2b) The function $x(s) = s$ is a exact solution of the equation $x'' + sx' - x = 0$.

(3a) The function $x(s) = s$ is a exact solution of the equation $x'' - sx' + x = 0$.

(3b) For $x'' - sx' - x = (x' - sx)' = 0$, then $x' - sx = 0$ gives $x = C_1 \exp\left(\frac{1}{2}s^2\right)$.

Then with Lemma 2 we obtain the conclusion of this theorem. $\square$

Proposition 1

The centroaffine arc length parameter of a curve x is also the Euclidean arc length parameter if and only if the Euclidean curvature function κ of x satisfies

$$\kappa\kappa'' - (\kappa')^2 + \kappa^4 + \varepsilon\kappa^2 = 0,$$

or

$$(\log \kappa)'' + \kappa^2 + \varepsilon = 0.$$

Therefore,

$$\kappa = 1 \quad \textit{and} \quad \varepsilon = -1,$$

or

$$\pm \int \frac{\mathrm{d}\kappa}{\kappa\sqrt{c_1 - \kappa^2 - 2\varepsilon \log \kappa}} = s + c_2 \quad (18)$$

where c_1 *and* c_2 *are integral constants.*

Curves in affine 2-space

Proof.

Assume that the centroaffine arc length parameter s of the curve $x(s)$ is also the Euclidean arc length parameter. From (12) we have

$$[x', x''] = [\alpha, \kappa\beta] = \varepsilon[x', x] = \varepsilon[\alpha, x],$$

that is

$$\kappa = \varepsilon[\alpha, x]. \tag{19}$$

Then

$$\kappa' = \varepsilon[\alpha', x] + \varepsilon[\alpha, x'] = \varepsilon\kappa[\beta, x], \tag{20}$$

$$\kappa'' = \varepsilon\kappa'[\beta, x] + \varepsilon\kappa[\beta', x] + \varepsilon\kappa[\beta, x'] = \varepsilon\kappa'[\beta, x] - \varepsilon\kappa^2[\alpha, x] + \varepsilon\kappa[\beta, \alpha]. \tag{21}$$

Therefore

$$\kappa'' = (\kappa')^2\kappa^{-1} - \kappa^3 - \varepsilon\kappa, \tag{22}$$

or

$$(\log\kappa)'' + \kappa^2 + \varepsilon = 0. \tag{23}$$

If $\kappa' = 0$, we know that $\varepsilon = -1$, $\kappa(s) = 1$ is the solution of (23). When $\kappa' \neq 0$, we have

$$2(\log\kappa)'(\log\kappa)'' = -2(\kappa^2 + \varepsilon)\frac{\kappa'}{\kappa} = -2\kappa\kappa' - 2\varepsilon\frac{\kappa'}{\kappa}.$$

$$\pm\int \frac{d\kappa}{\kappa\sqrt{c_1 - \kappa^2 - 2\varepsilon\log\kappa}} = s + c_2$$

where c_1 and c_2 are integral constants. □

(4) Centroaffine curves

in affine 3-space.

In this section we consider centroaffine curve $x = x(s) : \mathbf{I} \to \mathbb{A}^3$ with the centroaffine arc length s in the affine 3-space. From (14) we have

$$\kappa_1(s) = \frac{[e_3, e_2, x]}{[e_1, e_2, x]}, \kappa_2(s) = \frac{[e_1, e_3, x]}{[e_1, e_2, x]}. \tag{24}$$

If the curvature functions $\kappa_1(s)$ and $\kappa_2(s)$ satisfy

$$\kappa_1(s) : \kappa_2(s) = \text{constant} \neq 0,$$

or

$$\kappa_2(s) : \kappa_1(s) = \text{constant} \neq 0,$$

then we can get

$$x''' = ax'' + bx' + \varepsilon x, \qquad a, b \in \mathbf{R}. \quad (25)$$

According to the solutions of the cubic equation

$$f^3 - af^2 - bf - \varepsilon = 0 \tag{26}$$

we have the following conclusion.

Theorem 8

The centroaffine curve $x(s)$ with $a_1\kappa_1(s) + a_2\kappa_2(s) = 0$, $a_1a_2 \neq 0$, $a_1, a_2 \in \mathbf{R}$, can be written as one or an open part of the following

(a) $x(s) = e^{\lambda s}(1, s, s^2)$, $\lambda \neq 0$ *is the triple real root of* (26);

(b) $x(s) = (e^{\lambda_0 s}, e^{\mu_0 s}, se^{\mu_0 s})$, $\lambda_0 \neq \mu_0$, $\lambda_0\mu_0 \neq 0$, λ_0 *is the simple real root of* (26), μ_0 *is the double real root of* (26);

(c) $x(s) = (e^{\lambda_0 s}, e^{\lambda_1 s}, e^{\lambda_2 s})$, $\lambda_0\lambda_1\lambda_2 \neq 0$, $\lambda_0, \lambda_1, \lambda_2$ *are the three simple real roots of* (26);

(d) $x(s) = (e^{\lambda_0 s}, e^{\mu_1 s}\cos\mu_2 s, e^{\mu_1 s}\sin\mu_2 s)$, $\lambda_0\mu_1\mu_2 \neq 0$, $\mu_1, \mu_2 \in \mathbf{R}$, λ_0 *is the simple real root of* (26), $\mu_1 \pm i\mu_2$ *are the two complex roots of* (26).

Corollary 9

The centroaffine curve $x(s)$ with constant centroaffine curvature $\kappa_1(s)$ and constant centroaffine torsion $\kappa_2(s)$ can be written as one of the curves given by Theorem 8.

Proof.

When the centroaffine curve $x(s)$ has constant centroaffine curvature and constant centroaffine torsion, $x(s)$ satisfies (25) for $a = \kappa_2$ and $b = \kappa_1$. □

In the following, we discuss some special centroaffine curves in affine 3-space $\mathbb{A}^3$. At first, the centroaffine curvature functions satisfy $\kappa_1(s) = \pm s$, $\kappa_2(s) = 0$. We consider the following cases.

Case one: $\kappa_1 = \varepsilon s$.

In this case (13) becomes

$$x''' - \varepsilon s x' - \varepsilon x = 0, \qquad \varepsilon = \pm 1.$$

$$x'' - \varepsilon s x = C, \qquad \varepsilon = \pm 1, \qquad C \in \mathbb{A}^3.$$

The solutions of $f'' - \varepsilon s f = 0$ are

$$f = \sqrt{s} Z_{\frac{1}{3}} \left(i \sqrt{\varepsilon} \frac{2}{3} s^{\frac{3}{2}} \right).$$

Where $Z_\nu(s)$ is the cylinder function

$$Z_\nu = c_1 J_\nu + c_2 Y_\nu, \qquad c_1, c_2 \in \mathbf{R} \tag{27}$$

and $J_\nu(s)$ is the Bessel function of the first kind

$$J_\nu(s) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{s}{2}\right)^{\nu+2k}}{k!\Gamma(\nu+k+1)}; \tag{28}$$

$Y_\nu(s)$ is the Bessel function of the second kind

$$Y_\nu(s) = \frac{J_\nu(s)\cos\nu\pi - J_{-\nu}(s)}{\sin\nu\pi}. \tag{29}$$

A solution of

$$f'' - \varepsilon s f = c_3, \qquad c_3 \in \mathbf{R} \tag{30}$$

is

$$\begin{aligned} f_0 &= u \int \frac{W_1}{W} \mathrm{d}s + v \int \frac{W_2}{W} \mathrm{d}s \qquad (31) \\ &= -c_0 c_3 u \int v \mathrm{d}s + c_0 c_3 v \int u \mathrm{d}s \\ &= -c_0 c_3 \left(u \int v \mathrm{d}s - v \int u \mathrm{d}s \right). \end{aligned}$$

Where $c_0 \in \mathbf{R}$, u and v are the fundamental solutions of

$$a_2(s)f''(s) + a_1(s)f'(s) + a_0(s)f(s) = f'' - \varepsilon s f = 0; \quad (32)$$

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}; \quad W_1 = \begin{vmatrix} 0 & v \\ c_3 & v' \end{vmatrix}; \quad W_2 = \begin{vmatrix} u & 0 \\ u' & c_3 \end{vmatrix}.$$

$$W(u,v) = W(u(s_0), v(s_0)) \exp\left(-\int_{s_0}^{s} \frac{a_1(s)}{a_2(s)} \mathrm{d}s \right) = c_0^{-1}.$$

(cf. [P-Z], 0.2.1-6, p22)

Theorem 10

Let $x = x(s) : \mathbf{I} \to \mathbb{A}^3$ be a regular curve with the centroaffine arc length s and curvature functions $\kappa_1(s) = \varepsilon s$, $\kappa_2(s) = 0$ in the affine 3-space $\mathbb{A}^3$. Then the curve x can be written as the following

$$x(s) = C_1 f_0 + C_2 \mathrm{Re}\left\{\sqrt{s} Z_{\frac{1}{3}}\left(i\sqrt{\varepsilon}\frac{2}{3}s^{\frac{3}{2}}\right)\right\} + C_3 \mathrm{Im}\left\{\sqrt{s} Z_{\frac{1}{3}}\left(i\sqrt{\varepsilon}\frac{2}{3}s^{\frac{3}{2}}\right)\right\}, \tag{33}$$

where $C_1, C_2, C_3 \in \mathbb{A}^3$ and $f_0(s)$ is a solution of (30) given by (31).

Case two: $\kappa_1 = -\varepsilon s$.

In this case (13) becomes

$$x''' + \varepsilon s x' - \varepsilon x = 0, \quad \varepsilon = \pm 1.$$

Thus one of the solutions is s and

$$x'''' + \varepsilon s x'' = 0, \quad \varepsilon = \pm 1.$$

The solutions of $f'' + \varepsilon s f = 0$ are

$$f = \sqrt{s} Z_{\frac{1}{3}} \left(i \sqrt{-\varepsilon} \frac{2}{3} s^{\frac{3}{2}} \right).$$

Curves in affine 3-space

Therefore we have

Theorem 11

Let $x = x(s) : \mathbf{I} \to \mathbb{A}^3$ *be a regular curve with the centroaffine arc length* s *and curvature functions* $\kappa_1(s) = -\varepsilon s$, $\kappa_2(s) = 0$ *in the affine* 3*-space* $\mathbb{A}^3$*. Then the curve* $x(s)$ *can be written as the following*

$$C_1 s + C_2 \mathrm{Re} \int \left\{ \int \left[\sqrt{s} Z_{\frac{1}{3}} \left(i\sqrt{-\varepsilon} \frac{2}{3} s^{\frac{3}{2}} \right) \right] \mathrm{d}s \right\} \mathrm{d}s \qquad (34)$$
$$+ C_3 \mathrm{Im} \int \left\{ \int \left[\sqrt{s} Z_{\frac{1}{3}} \left(i\sqrt{-\varepsilon} \frac{2}{3} s^{\frac{3}{2}} \right) \right] \mathrm{d}s \right\} \mathrm{d}s,$$

where $C_1, C_2, C_3 \in \mathbb{A}^3$.

Remark

The centroaffine arc length parameter of the curves given in Theorem 10 *and Theorem* 11 *is also equiaffine arc length parameter.*

(5) Curves

in Euclidean 3-space.

C in Euclidean 3-space

Let $x = x(s) : \mathbf{I} \to \mathbb{A}^3$ be a regular curve with the centroaffine arc length s in the affine 3-space $\mathbb{A}^3$. We consider also $\mathbb{A}^3$ as the Euclidean 3-space with the standard inner product.

C in Euclidean 3-space

If the centroaffine arc length parameter is also the Euclidean arc length parameter of the curve $x(s)$, we have

$$
\begin{aligned}
x \cdot \gamma &= \varepsilon \kappa \tau, \qquad \varepsilon = \pm 1, \\
x \cdot \beta &= -\frac{(\varepsilon \kappa \tau)'}{\tau}, \\
x \cdot \alpha &= \varepsilon \tau^2 + \frac{\varepsilon}{\kappa} \left(\frac{(\kappa \tau)'}{\tau} \right)' .
\end{aligned}
$$

Therefore, the centroaffine arc length parameter is also the Euclidean arc length parameter, if and only if

$$1 - \frac{\varepsilon\kappa}{\tau}(\kappa\tau)' = \varepsilon\left(\frac{\tau}{\kappa}(\kappa\tau) + \frac{1}{\kappa}\left(\frac{(\kappa\tau)'}{\tau}\right)'\right)' \qquad (35)$$

or

$$\begin{aligned}\varepsilon &= 2\tau\tau' + [(\log\kappa)'' + \kappa^2](\log\kappa\tau)' \\ &\quad + (\log\kappa)'(\log\kappa\tau)'' + (\log\kappa\tau)'''.\end{aligned} \qquad (36)$$

Proposition 2

Let $x = x(s) : \mathbf{I} \to \mathbb{A}^3$ be a regular curve with the centroaffine arc length s in the affine 3-space $\mathbb{A}^3$. Then s is also the Euclidean arc length parameter of the curve $x(s)$ if and only if the curvature function $\kappa(s)$ and torsion function $\tau(s)$ of $x(s)$ satisfy (35) or (36).

In the following we give two examples of the solutions of (35) or (36).

Example 1

Let $x(s)$ be a regular curve with the Euclidean arc length parameter s and the curvature function $\kappa(s) = as^{-\frac{1}{2}}$, torsion function $\tau(s) = \pm\sqrt{s} = \varepsilon_0\sqrt{s}$, $\varepsilon_0 = \pm 1$, $a > 0$. Then from [C] we know that the curve is the rectifying curve. It is easy to check that $\kappa(s)$ and $\tau(s)$ are solutions of equation (36) for $\varepsilon = 1$.

As the curve in Euclidean space, using Frenet formulas, by a direct calculation we have

$$x'''' = -\frac{1}{2}s^{-1}x''' + (\frac{1}{2}s^{-2} - a^2s^{-1} - s)x'' + a^2s^{-2}x'.$$

On the other hand, from (24) we have

$$\kappa_1 = -a^2s^{-1} - s, \qquad \kappa_2 = -\frac{1}{2}s^{-1}. \quad (37)$$

Then (13) becomes

$$sx''' + \frac{1}{2}x'' + (s^2 + a^2)x' - sx = 0. \quad (38)$$

Therefore, using the characterization of the rectifying curve, x can be written as

$$x(t) = \frac{b}{\cos t} x_0(t),$$

where $x_0^2(t) = 1$, $\left(\frac{\mathrm{d}x_0}{\mathrm{d}t}\right)^2 = 1$, $s = b \tan t$,

$$b\kappa = (\cos^3 t)\kappa_g,$$

where κ_g *is the spherical curvature function of* $x_0(t)$*. The tangent vector* α *of* x *and tangent vector* α_0 *of* x_0 *satisfy*

$$\alpha = (\sin t)x_0 + (\cos t)\alpha_0.$$

The spherical curve $x_0(t)$ *satisfies*

$$x_0''' + (1 + \kappa_g^2)x_0' - \frac{\kappa_g'}{\kappa_g}(x_0'' + x_0) = 0.$$

C in Euclidean 3-space

Example 2

We now consider a spherical curve $x(s)$ *in Euclidean space with the Euclidean arc length parameter* s *and spherical curvature function* $\kappa_g(s)$.

For $\kappa_g' = \varepsilon \kappa_g$, *that is* $\kappa_g = e^{\varepsilon s}$, *we have*

$$\kappa = \sqrt{1+\kappa_g^2} = \sqrt{1+e^{2\varepsilon s}},$$

$$\tau = \frac{\pm \kappa_g'}{1+\kappa_g^2} = \frac{\pm \varepsilon e^{\varepsilon s}}{1+e^{2\varepsilon s}}.$$

They are also the solutions of (36). The curve is a spherical curve with the spherical curvature function $\kappa_g = e^{\varepsilon s}$.

And from (24) we have

$$\begin{cases} \kappa_1 = -(1+\kappa_g^2) \\ \quad\; = -(1+e^{2\varepsilon s}), \\ \kappa_2 = \dfrac{\kappa_g'}{\kappa_g} = \varepsilon. \end{cases} \tag{39}$$

Then (13) becomes

$$x''' - \varepsilon x'' + (1+e^{2\varepsilon s})x' - \varepsilon x = 0. \tag{40}$$

(6) Curves

in lightlike cone

of Minkowski 3-space.

Curves in lightlike cone

Let $x = x(s) : \mathbf{I} \to \mathbb{A}^3$ be a regular curve with the centroaffine arc length s in the affine 3-space $\mathbb{A}^3$. We consider $\mathbb{A}^3$ also as the Minkowski 3-space with the standard Minkowski inner product.

If x is also a cone curve in Minkowski 3-space, from (11) we have

$$\mathrm{d}s = e^{\sigma}\mathrm{d}s_m = (\kappa_g')^{\frac{1}{3}}\mathrm{d}s_m,$$

where s_m is the arc length parameter of cone curves in Minkowski 3-space.

The centroaffine arc length parameter is also the (affine) arc length parameter of cone curve if and only if

$$\kappa_g' = \text{constant}.$$

Cone curvature function $\kappa_g(s) = \varepsilon s = \pm s$, s is also equiaffine and centroaffine arc length parameter.

When the case $\kappa_g = \varepsilon s$, from the structure equations of cone curve we have

$$x''' - 2\varepsilon s x' - \varepsilon x = 0, \quad \varepsilon = \pm 1.$$

The centroaffine curvature $\kappa_1 = 2\kappa_g$ and the centroaffine torsion $\kappa_2 = 0$.

Solving this equation we have (see [P-Z], 3.1.2-2.48, p499)

$$x = C_1 u^2 + C_2 uv + C_3 v^2,$$

$C_1, C_2, C_3 \in \mathbb{A}^3$ and x can be written as

$$x = (u^2 - v^2, 2uv, u^2 + v^2).$$

Where u and v are the solutions of

$$2f'' - \varepsilon s f = 0.$$

Curves in lightlike cone

That is $f(s) = u(s) + iv(s) =$

$$\sqrt{s}Z_{\frac{1}{3}}\left(i\sqrt{\frac{\varepsilon}{2}\frac{s^{\frac{3}{2}}}{\frac{3}{2}}}\right) = \sqrt{s}Z_{\frac{1}{3}}\left(i\sqrt{\varepsilon}\frac{\sqrt{2}}{3}s^{\frac{3}{2}}\right).$$

The cylinder function $Z_\nu(s)$ is defined by (27). Bessel function of the first kind $J_\nu(s)$ is defined by (28). Bessel function of the second kind $Y_\nu(s)$ is defined by (29).

Therefore we get

Theorem 13

Let $x = x(s) : \mathbf{I} \to \mathbb{A}^3$ *be a regular curve with the centroaffine arc length* s*, centroaffine curvature function* $\kappa_1 = 2\varepsilon s$ *and torsion function* $\kappa_2 = 0$ *in the affine 3-space* $\mathbb{A}^3$.

Then it is a cone curve in Minkowski 3-space with the arc length parameter s and can be written as

$$x = (u^2 - v^2, 2uv, u^2 + v^2) \qquad (41)$$

and where

$$u(s) + iv(s) = \sqrt{s} Z_{\frac{1}{3}} \left(i\sqrt{\varepsilon} \frac{\sqrt{2}}{3} s^{\frac{3}{2}} \right). \qquad (42)$$

When $\kappa_g = a s_m^{-2}$ we have $\mathrm{d}s = (b s_m)^{-1} \mathrm{d}s_m$, that is $e^{bs} = s_m$, here $b^{-3} = -2a$. Then from (24), we have

$$\begin{aligned} \kappa_1(s) &= 2ab^2 - 2b^2, \\ \kappa_2(s) &= 3b. \end{aligned} \tag{43}$$

Therefore from (13) we have

$$x''' - 3bx'' - (2ab^2 - 2b^2)x' - \varepsilon x = 0.$$

The solutions of this equation can be written as (see also [LIU-Curves])

$$
\begin{aligned}
x(s) &= C_1 s_m + C_2 s_m^{(1+\sqrt{1+2a})} + C_3 s_m^{(1-\sqrt{1+2a})} \\
&= C_1 e^{bs} + C_2 e^{b(1+\sqrt{1+2a})s} + C_3 e^{b(1-\sqrt{1+2a})s}
\end{aligned} \qquad (44)
$$

for $a > -\frac{1}{2}$, where C_1, C_2, $C_3 \in \mathbb{A}^3$,

$$
\begin{aligned}
x(s) &= C_1 s_m + C_2 s_m \sin[(\sqrt{-1-2a}) \log s_m] \\
&\quad + C_3 s_m \cos[(\sqrt{-1-2a}) \log s_m] \\
&= C_1 e^{bs} + C_2 e^{bs} \sin[b(\sqrt{-1-2a}) s] \\
&\quad + C_3 e^{bs} \cos[b(\sqrt{-1-2a}) s]
\end{aligned}
\tag{45}
$$

for $a < -\frac{1}{2}$, where $C_1, C_2, C_3 \in \mathbb{A}^3$, and

$$
\begin{aligned}
x(s) &= C_1 s_m + C_2 s_m \log s_m + C_3 s_m \log^2 s_m \\
&= C_1 e^{bs} + C_2 (bs) e^{bs} + C_3 (bs)^2 e^{bs}
\end{aligned}
\tag{46}
$$

for $a = -\frac{1}{2}$, where $C_1, C_2, C_3 \in \mathbb{A}^3$.

The curves (44), (45) and (46) can be written as (see [Liu-1], [L-M])

$$x = \frac{e^{bs}}{2c}(e^{bcs} - e^{-bcs}, 2, e^{bcs} + e^{-bcs}), \tag{47}$$

$$x = \frac{s\sin(cbs)}{2c}\left(\frac{2}{c}\tan\left(\frac{cbs}{2}\right) - \frac{c}{2}\tan^{-1}\left(\frac{cbs}{2}\right), 2, \frac{2}{c}\tan\left(\frac{cbs}{2}\right) + \frac{c}{2}\tan^{-1}\left(\frac{cbs}{2}\right)\right), \tag{48}$$

and

$$x = \frac{bse^{bs}}{c}\left(\frac{c}{bs} - \frac{bs}{c}, 2, \frac{c}{bs} + \frac{bs}{c}\right). \tag{49}$$

Theorem 14

Let $x = x(s) : \mathbf{I} \to \mathbb{A}^3$ be a regular curve with the centroaffine arc length s, centroaffine curvature function $\kappa_1 = 2ab^2 - 2b^2$ and torsion function $\kappa_2 = 3b$ in the affine 3-space $\mathbb{A}^3$. Then it can be written as (47), (48) or (49) and is a cone curve in Minkowski 3-space.

Proposition 3

The centroaffine arc length parameter of a curve x in affine 2-space $\mathbb{A}^2$ is also the Minkowski arc length parameter if and only if the Minkowski curvature function κ of x satisfies

$$(\log \kappa)'' - \kappa^2 + \varepsilon = 0$$

for the spacelike curve and also timelike curve.

Proposition 4

The centroaffine arc length parameter of a curve x in affine 3-space $\mathbb{A}^3$ is also the Minkowski arc length parameter if and only if the Minkowski curvature function κ and torsion function τ of x satisfies

$$\begin{aligned} \varepsilon &= -2\tau\tau' + [(\log\kappa)'' + \kappa^2](\log\kappa\tau)' \\ &\quad + (\log\kappa)'(\log\kappa\tau)'' + (\log\kappa\tau)''' \end{aligned} \tag{50}$$

for the first kind of spacelike curve;

Continue

$$\varepsilon = -2\tau\tau' + [(\log\kappa)'' - \kappa^2](\log\kappa\tau)' + (\log\kappa)'(\log\kappa\tau)'' + (\log\kappa\tau)''' \tag{51}$$

for the second kind of spacelike curve;

$$\varepsilon = 2\tau\tau' + [(\log\kappa)'' - \kappa^2](\log\kappa\tau)' + (\log\kappa)'(\log\kappa\tau)'' + (\log\kappa\tau)'''. \tag{52}$$

for the timelike curve;

$$\varepsilon = -2\tau\kappa' - (\kappa\tau)' + \left(\frac{\kappa''}{\kappa}\right)' \tag{53}$$

for the null curve.

Paper and Preprint

[1] Huili Liu, *Curves in affine and Semi-Euclidean spaces*, Results in Mathematics, **65**(2014), 235-249.

[2] *Centroaffine curves in affine space*. Preprint.

END

Continue

- curves in de Sitter space or hyperbolic space;
- curves in space forms;
- curves and surfaces;
- relations between curves and surfaces;
- ....

References

- [C] B. Y. Chen, *When does the position vector of a space curve always lie in its rectifying plane*, Amer. Math. Monthly, **110**(2003), 147-152.
- [LIU-Curves] Huili Liu, *Curves in the lightlike cone*, Contrib. Algebr. Geom., **45**(2004), 291-303. Erratum: **54**(2013), 469-470.

References

- [Liu-1] Huili Liu, *Ruled surfaces with lightlike ruling in 3-Minkowski space*, Journal of Geometry and Physics, **59**(2009), 74-78.
- [L-M] Huili Liu, Qingxian Meng, *Representation formulas of curves in a 2 and 3 dimensional lightlike cone*, Results Math., **59**(2011), 437-451. Erratum: **61**(2012), 423-424.

References

- [P-Z] A. D. Polyanin, V. F. Zaitsev *Handbook of Exact Solutions for Ordinary Differential Equations*, second edition, 2003, Chapman & Hall/CRC.
- [L-S-Z] A. M. Li, U. Simon, G. Zhao, *Global Affine Differential Geometry of Hypersurfaces*, W. De Gruyter, Berlin-New York, 1993.

[S-S-V] U. Simon, A. Schwenk-Schellschmidt, H. Viesel, *Introduction to the Affine Differential Geometry of Hypersurfaces*, Lecture Notes, Science University Tokyo, ISBN 3-7983-1529-9, 1991.

Thank you for your attention!

Vielen Dank!

谢谢！