Symmetrization with respect to Mixed Volumes

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Interaction Between PDEs and Convex Geometry

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Schwarz symmetrization

- Ω an open bounded set of $\mathbb{R}^n$
  $u : \Omega \to \mathbb{R}$ a Borel measurable function
- Symmetric rearrangement:
  the centered open ball having the same volume as Ω,

$$\Omega^\sharp = B_R(0), \text{ where } \omega_n R^n = \mathrm{Vol}(\Omega).$$

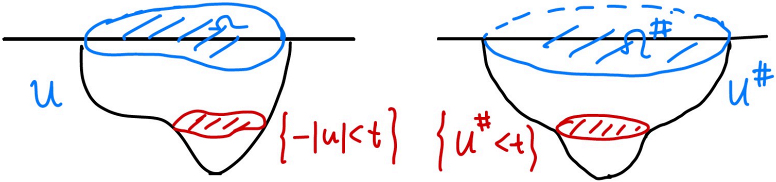
- Schwarz Symmetrization (symmetric decreasing rearrangement) for u:

$$\begin{aligned} u^\sharp : \quad & \Omega^\sharp \to \mathbb{R}, \\ & u^\sharp(x) = u^\sharp(|x|) = \sup\{t < 0 : \{-|u| < t\} \leq \omega_n |x|^n\}. \end{aligned}$$

- By definition,

$$\mathrm{Vol}(\{u^\sharp < t\}) = \mathrm{Vol}(\{-|u| < t\}).$$

Schwarz symmetrization



Schwarz symmetrization

Properties of Schwarz symmetrization: for $u \in W^{1,p}$,

- (Cavalieri's principle)

$$\int_{\Omega} |u|^p dx = \int_{\Omega^\sharp} |u^\sharp|^p dx.$$

- (Pólya-Szegö's principle)

$$\int_{\Omega} |\nabla u|^p dx \geq \int_{\Omega^\sharp} |\nabla u^\sharp|^p dx.$$

- (Hardy-Littlewood's inequality)

$$\int_{\Omega} |u||v| dx \leq \int_{\Omega^\sharp} u^\sharp v^\sharp dx.$$

Schwarz symmetrization

Proof of Cavalieri's principle:
By Layer cake formula,

$$\begin{aligned}\int_\Omega |u|^p dx &= \int_{-\infty}^0 p(-t)^{p-1}\mathrm{Vol}(\{-|u| < t\})dt \\ &= \int_{-\infty}^0 p(-t)^{p-1}\mathrm{Vol}(\{u^\sharp < t\})dt \\ &= \int_{\Omega^\sharp} |u^\sharp|^p dx.\end{aligned}$$

Schwarz symmetrization

Proof of Pólya-Szegö's principle:
By co-area formula,

$$\int_\Omega |\nabla u|^2 dx = \int_{-\sup|u|}^{-\inf|u|} \int_{\{-|u|=t\}} |\nabla u| d\mu dt,$$
$$\frac{d}{dt}\mathrm{Vol}(\{-|u|<t\}) = \int_{\{-|u|=t\}} \frac{1}{|\nabla u|} d\mu.$$

Since $\mathrm{Vol}(\{-|u|<t\}) = \mathrm{Vol}(\{u^\sharp < t\})$, by isoperimetric inequality,

$$\mathrm{Area}(\{-|u|=t\}) \geq \mathrm{Area}(\{u^\sharp = t\}), \quad \int_{\{-|u|=t\}} \frac{1}{|\nabla u|} d\mu = \int_{\{u^\sharp=t\}} \frac{1}{|\nabla u^\sharp|} d\mu.$$

By Hölder's inequality,

$$\mathrm{Area}(\{-|u|=t\}) \leq \int_{\{-|u|=t\}} |\nabla u| d\mu \int_{\{-|u|=t\}} \frac{1}{|\nabla u|} d\mu,$$
$$\mathrm{Area}(\{u^\sharp=t\}) = \int_{\{u^\sharp=t\}} |\nabla u^\sharp| d\mu \int_{\{u^\sharp=t\}} \frac{1}{|\nabla u^\sharp|} d\mu.$$

Schwarz symmetrization

Applications of Schwarz symmetrization I:

- (Rayleigh-Faber-Krahn's inequality for first Dirichlet eigenvalue)

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^\sharp).$$

- (Saint-Venant's principle for torsional rigidity)

$$\tau(\Omega) \leq \tau(\Omega^\sharp).$$

- Because of the variational property

$$\lambda_1(\Omega) = \inf_{u \neq 0} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2}, \quad \tau(\Omega) = \sup_{u \neq 0} \frac{\left(\int_\Omega u\right)^2}{\int_\Omega |\nabla u|^2}$$

Schwarz symmetrization

Applications of Schwarz symmetrization II:

- (Talenti '76) Let u be the solution to

  $$\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

  and

  $$\Delta v = f^\sharp \text{ in } \Omega^\sharp, \quad v = 0 \text{ on } \partial\Omega^\sharp,$$

  Then

  $$|u^\sharp| \le v \text{ in } \Omega^\sharp.$$

- Because $\sup |u| = \sup |u^\sharp|$, this gives a sharp estimate for $|u|$.

Tatenti-Tso symmetrization

- $\Omega \subset \mathbb{R}^n$ a bounded, convex domain with C^2 boundary
- Steiner's formula:

$$\mathrm{Vol}\,(\Omega + tB) = \sum_{k=0}^{n} \binom{n}{k} t^k W_k(\Omega),$$

where $W_k(\Omega)$ is k-th quermassintegral given by

$$W_0(\Omega) = \mathrm{Vol}(\Omega), \quad W_k(\Omega) = \frac{1}{n} \int_{\partial\Omega} H_{k-1}(\kappa) d\mathcal{H}^{n-1}.$$

H_k are k-th mean curvature.
Denote $\zeta_k(\Omega)$ k-mean radius given by

$$\zeta_k(\Omega) = \left(\frac{W_k(\Omega)}{\omega_n} \right)^{\frac{1}{n-k}}.$$

- Alexandrov-Fenchel's inequality for quermassintegral:

$$\zeta_k(\Omega) \geq \zeta_l(\Omega), \quad \text{“} = \text{” iff } \Omega \text{ is a ball.}$$

Tatenti-Tso symmetrization w.r.t. Quermassintegral

- Set of admissible functions

$$\Phi_0(\Omega) = \left\{u \in C^2(\bar{\Omega}) : u = 0 \text{ on } \partial\Omega, u \text{ strictly convex}\right\}.$$

- Let $\Omega^\sharp_{k-1}$ be the centered open ball having the same W_k as Ω, i.e.

$$\Omega^\sharp_{k-1} = B_R(0), \text{ where } R = \zeta_k(\Omega).$$

- $(k-1)$-symmetrand of u:

$$u^\sharp_{k-1}(x) = \sup\left\{t \leq 0 : \zeta_{k-1}(\{u < t\}) \leq |x|\right\}.$$

It follows

$$\zeta_{k-1}(\{u < t\}) = \zeta_{k-1}(\{u^\sharp_{k-1} < t\}).$$

- The case $k = 1$ is Schwarz symmetrization.

Talenti-Tso symmetrization

- $k = 1, 2, \cdots, n$. The k-Hessian integral

$$I_k[u, \Omega] = \int_\Omega (-u)\sigma_k(\nabla^2 u)dx$$

$k = 1$ Dirichlet integral,
$k = n$ Monge-Ampere integral.

Theorem (Talenti '81 $n = k = 2$ and Tso '89 any n and k)
For $u \in \Phi_0(\Omega)$,

$$I_k[u, \Omega] \geq I_k[u^\sharp_{k-1}, \Omega^\sharp_{k-1}].$$

Equality holds if and only if Ω is a ball and u is radial.

- The proof used crucially
  Alexandrov-Fenchel's inequality for quermassintegrals
  Reilly's work on Hessian operators

Convex symmetrization

- Let F be a norm on $\mathbb{R}^n$, i.e., positive, even, convex and 1-homogenous.
- Let F^0 be its dual norm, i.e.,

$$F^0(\xi) = \sup_{x \neq 0} \frac{\langle x, \xi \rangle}{F(\xi)}.$$

- $\mathcal{W}_F = \{F^0(\xi) \leq 1\}$ unit Wulff ball, $\partial \mathcal{W}_F$ unit Wulff shape
  Denote $\kappa_n = \mathrm{Vol}(\mathcal{W}_F)$ and $\mathcal{W}_R$ the centered Wulff ball with radius R.
- Anisotropic Dirichlet integral

$$\int_\Omega F(\nabla u)^2$$

- Anisotropic Laplacian

$$\Delta_F u = \mathrm{div}(\nabla_\xi(\frac{1}{2}F^2)(\nabla u))$$

Convex symmetrization

Alvino-Ferone-Lions-Trombetti '97 introduces convex symmetrization which diminishes the anisotropic Dirichlet integral

- Convex symmetrization of Ω

$$\Omega^* = \mathcal{W}_R, \text{ where } \mathrm{Vol}(\mathcal{W}_R) = \kappa_n R^n = \mathrm{Vol}(\Omega).$$

Convex symmetrization of u:

$$u^* : \Omega^* \to \mathbb{R},$$
$$u^*(x) = u^*(F^0(x)) = \sup\{t < 0 : |\{-|u| < t\}| \leq \kappa_n (F^0(x))^n\}.$$

Convex symmetrization

Theorem (Alvino-Ferone-Lions-Trombetti '97)

For $u \in W_0^{1,2}(\Omega)$,

$$\int_\Omega F(\nabla u)^2 \geq \int_{\Omega^*} F(\nabla u^*)^2.$$

Equality holds iff Ω *is a Wulff ball and* u *is radial w.r.t.* F*, namely,* $u(x) = u(F^0(x))$.

Corollary

Let u *be the solution to*

$$\Delta_F u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

and

$$\Delta_F v = f^* \text{ in } \Omega^*, \quad v = 0 \text{ on } \partial\Omega^*,$$

Then

$$|u^*| \leq v \text{ in } \Omega^*.$$

Symmetrization w.r.t. mixed volumes

- Our aim is to study Talenti-Tso symmetrization in the anisotropic case.
- Motivation: Alexandrov-Fenchel's inequality for mixed volumes of two convex bodies

  To be precise, Let F be a given norm whose Wulff ball is given by $\mathcal{W}_F$, Let Ω be a convex domain with C^2 boundary, then

$$\mathrm{Vol}((1-t)\Omega + t\mathcal{W}_F) = \sum_{k=0}^{n} \binom{n}{k} (1-t)^{n-k} t^k W_k(\Omega, \mathcal{W}_F).$$

  $W_k(\Omega, \mathcal{W}_F)$ has differential geometric representation

$$W_0(\Omega, \mathcal{W}_F) = \mathrm{Vol}(\Omega), \quad W_k(\Omega, \mathcal{W}_F) = \frac{1}{n} \int_{\partial\Omega} H_{k-1,F} d\mathcal{H}^{n-1}.$$

  Denote

$$\zeta_{k,F}(\Omega) = \left(\frac{W_k(\Omega, \mathcal{W}_F)}{\kappa_n} \right)^{\frac{1}{n-k}}$$

  Then

$$\zeta_{k,F}(\Omega) \geq \zeta_{l,F}(\Omega), \text{ for } 0 \leq l < k \leq n-1,$$

  Equality holds iff Ω is a Wulff ball.

Symmetrization w.r.t. mixed volumes

- Let $F \in C^2(\mathbb{R}^n \setminus \{0\})$ be a strongly convex norm on $\mathbb{R}^n$, strongly convex means $\mathrm{Hess}(\frac{1}{2}F^2)$ is positive definite.
- Denote by $A_F[u] = ((A_F)_{ij}[u])$ the matrix

$$
\begin{aligned}
(A_F)_{ij}[u] &= \partial_{x_j}\left[\partial_{\xi_i}\left(\frac{1}{2}F^2\right)(\nabla u)\right] \\
&= \sum_l \left(\frac{1}{2}F^2\right)_{il}(\nabla u)u_{lj}, \text{ when } \nabla u \neq 0.
\end{aligned}
$$

  We regard $A_F[u] = 0$ when $\nabla u = 0$, in the case that F is not the Euclidean norm.
- In case F is the Euclidean norm, $A_F[u] = \nabla^2 u$.

Symmetrization w.r.t. mixed volumes

- The anisotropic k-Hessian operator of u is defined as

$$S_{k,F}[u] = S_k(A_F[u]).$$

- The anisotropic k-Hessian integral of u is defined by

$$\begin{aligned} I_{k,F}[u,\Omega] &= \int_\Omega (-u)S_{k,F}[u]\,dx = \int_\Omega (-u)S_k(A_F[u])\,dx \\ &= \int_\Omega S^{ij}_{k,F}[u]FF_iu_j\,dx. \text{ (when } u|_{\partial\Omega} = 0) \end{aligned}$$

  The second line follows from $\partial_j S^{ij}_{k,F} = 0$.
- In case F is the Euclidean norm,

$$A_F[u] = \nabla^2 u, \quad S_{k,F}[u] = S_k(\nabla^2 u), \quad I_{k,F}[u,\Omega] = I_k[u,\Omega].$$

Symmetrization w.r.t. mixed volumes

- Let Ω^*_{k-1} be the centered open ball having the same $W_{k,F}$ as Ω, i.e.

$$\Omega^*_{k-1} = \mathcal{W}_R, \text{ where } R = \zeta_{k,F}(\Omega).$$

For $u \in \Phi_0(\Omega) = \{u \in C^2(\bar{\Omega}) : u = 0 \text{ on } \partial\Omega, u \text{ strictly convex}\}$,

$$u^*_{k-1}(x) = \sup\left\{t \leq 0 : \zeta_{k-1,F}(\{u < t\}) \leq F^0(x)\right\}.$$

It follows

$$\zeta_{k-1,F}(\{u < t\}) = \zeta_{k-1,F}(\{u^*_{k-1} < t\}).$$

Symmetrization w.r.t. mixed volumes

- Pólya-Szegö type principle

Theorem (Della Pietra-Gavitone '15, Della Pietra-Gavitone-X. '21)

For $u \in \Phi_0(\Omega)$,

$$\int_\Omega |u|^p dx \leq \int_{\Omega^*_{k-1}} |u^*_{k-1}|^p dx,$$

$$I_{k,F}[u, \Omega] \geq I_{k,F}[u^*_{k-1}, \Omega^*_{k-1}].$$

Equality holds iff Ω *is a Wulff ball and* u *is radial w.r.t.* F*, namely,* $u(x) = u(F^0(x))$.

- Della Pietra-Gavitone '15 proved the case $n = k = 2$ by direct computation.
- Difficulty for general case, compare to the case of Euclidean norm, is the study of k-Hessian operator S_k on non-symmetric matrix $A_F[u]$.

Symmetrization w.r.t. mixed volumes

- Define anisotropic L^p $k-$Hessian integral

$$I_{k,p,F}[u,\Omega] = \int_\Omega S_k^{ij}[u]F^{p-k}F_i u_j \, dx.$$

In particular,

$$I_{k,k+1,F} = k\, I_{k,F}, \quad I_{1,p,F} = \int_\Omega F^p(\nabla u)\, dx.$$

Theorem (Della Pietra-Gavitone-X. '21)
For $u \in \Phi_0(\Omega)$, $p \geq 1$,

$$I_{k,p,F}[u,\Omega] \geq I_{k,p,F}[u^*_{k-1}, \Omega^*_{k-1}].$$

Equality holds iff Ω is a Wulff ball and u is radial w.r.t. F, namely, $u(x) = u(F^0(x))$.

Symmetrization w.r.t. mixed volumes

Corollary (Anisotropic Sobolev type inequality with optimal constant)

For $u \in \Phi_0(\Omega)$,

- *if* $p < n-k+1$, *then*

$$\|u\|^p_{L^{\frac{np}{n-k+1-p}}(\Omega)} \leq C(n,k,p,F) I_{k,p,F}[u,\Omega],$$

- *if* $p > n-k+1$, *then*

$$\|u\|^p_{L^\infty(\Omega)} \leq C(n,k,p,F) I_{k,p,F}[u,\Omega],$$

- *if* $p = n-k+1$, *then*

$$\|u\|^p_{L^\Psi(\Omega)} \leq C I_{p,k,F}[u;\Omega].$$

where $L^\Psi(\Omega)$ *is the Orlicz space associated to the function* $\Psi(t) = e^{|t|^{\frac{p}{p-1}}} - 1$.

Symmetrization w.r.t. mixed volumes

Theorem (A priori estimate for anisotropic Hessian equation)
Let $u \in \Phi_0(\Omega)$ be a solution of the following Dirichlet problem

$$\begin{cases} S_{k,F}[u] = f(x) \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then

$$u^*_{k-1} \geq v \text{ in } \Omega^*_{k-1},$$

where v is the unique anisotropic radially symmetric solution of the following symmetrized problem:

$$\begin{cases} S_{k,F}[v] = f^*_{0,F}(x) & \text{in } \Omega^*_{k-1,F} \\ v = 0 & \text{on } \Omega^*_{k-1,F}. \end{cases}$$

Important ingredients

- A study of S_k for non-symmetric matrix

$$S_k(A) = \frac{1}{k!} \sum_{1 \le i_1, \cdots, i_k, j_1, \cdots, j_k \le n} \delta_{i_1 \cdots i_k}^{j_1 \cdots j_k} A_{i_1 j_1} \cdots A_{i_k j_k},$$

$$S_k^{ij}(A) = \frac{\partial S_k(A)}{\partial A_{ij}}$$

$$= \frac{1}{(k-1)!} \sum_{1 \le i_1, \cdots, i_k, j_1, \cdots, j_k \le n} \delta_{i_1 \cdots i_{k-1} i}^{j_1 \cdots j_{k-1} j} A_{i_1 j_1} \cdots A_{i_{k-1} j_{k-1}}.$$

Important ingredients

Proposition

For an $n \times n$ matrix $A = (A_{ij})$, we have

$$S_k^{ij}(A) = S_{k-1}(A)\delta_{ij} - \sum_l S_{k-1}^{il}(A)A_{jl}.$$

Proposition

$$\sum_j \partial_j S_{k,F}^{ij}[u] = 0.$$

Important ingredients

- A study of anisotropic Hessian integral on anisotropic radial function

Proposition

Let $u(x) = v(r)$, where $r = F^0(x)$. Then

$$S_{k,F}[u] = \binom{n-1}{k-1}\frac{v''(r)}{r}\left(\frac{v'(r)}{r}\right)^{k-1} + \binom{n-1}{k}\left(\frac{v'(r)}{r}\right)^{k}$$
$$= \binom{n-1}{k-1} r^{-(n-1)}\left(\frac{r^{n-k}}{k}(v'(r))^k\right)'.$$

$$I_{k,F}[u, \mathcal{W}_R] = \kappa_n \binom{n}{k}\int_0^R r^{n-k} v'(r)^{k+1}\, dr.$$

Important ingredients

- A study of anisotropic curvatures of level sets
- Let M be a smooth closed hypersurface in $\mathbb{R}^n$ and ν be the unit Euclidean outer normal of M. The anisotropic outer normal of M is defined by
$$\nu_F = \nabla F(\nu).$$
The anisotropic principal curvatures $\kappa_F = (\kappa_1^F, \ldots, \kappa_{n-1}^F) \in \mathbb{R}^{n-1}$ are defined as the eigenvalues of the map
$$d\nu_F \colon T_pM \to T_{\nu_F(p)}\mathcal{W}_F.$$
For $k = 1, \ldots, n$ the anisotropic k-th mean curvature of M is
$$H_F = \frac{1}{\binom{n-1}{k}} \sigma_k(\kappa_F).$$

Important ingredients

Proposition

Assume Σ_t is a non-degenerate level set of u, i.e., $\nabla u \neq 0$ along Σ_t. Then the anisotropic k-th mean curvature $\sigma_k(\kappa_F)$ of Σ_t

$$H_{k,F} = \frac{1}{\binom{n-1}{k}} S_k\left(\sum_l F_{il}u_{lj}\right) = \frac{1}{\binom{n-1}{k}} \frac{1}{F^{k+1}} \sum_{i,j} S^{ij}_{k+1,F}[u]u_jF_i,$$

- In the case F is the Euclidean norm, it reduces to (Reilly '70s)

$$H_k = \frac{1}{\binom{n-1}{k}} \sum_{i,j} \frac{S^{ij}_{k+1}(\nabla^2 u)u_iu_j}{|\nabla u|^{k+2}},$$

- In the case $k = 1$, it reduces to (Wang-X. '11)

$$H_F = \sum_{i,j} F_{ij}u_{ij} = \frac{1}{F}\left(\Delta_F u - \sum_{i,j} F_iF_ju_{ij}\right).$$

Important ingredients

Proposition (Reilly '70s, He-Li '08)

For regular sublevel set $\Omega_t = \{u < t\}$,

$$\frac{d}{dt} W_{k,F}(\overline{\Omega_t}) = \frac{1}{\binom{n}{k}} \int_{\Sigma_t} \frac{S_k(\kappa_F) F(\nu)}{F(\nabla u)} d\mathcal{H}^{n-1}.$$

$$\frac{d}{dt} \zeta_{k,F}(\overline{\Omega_t}) = \frac{1}{(n-k)\kappa_n \binom{n}{k}} \frac{1}{[\zeta_{k,F}(\overline{\Omega_t})]^{n-k-1}} \int_{\Sigma_t} \frac{S_k(\kappa_F) F(\nu)}{F(\nabla u)} d\mathcal{H}^{n-1}.$$

Sketch of proof

By Alexandrov-Fenchel, variational formula, and Hölder inequality,

$$
\begin{aligned}
& [\zeta_{k-1}(\overline{\Omega_t})]^{(n-k)(k+1)} \\
\leq \;& [\zeta_k(\overline{\Omega_t})]^{(n-k)(k+1)} \\
= \;& C_{n,k}\left(\int_{\Sigma_t} H_{k-1,F} F(\nu) d\mathcal{H}^{n-1}\right)^{k+1} \\
\leq \;& C_{n,k}\left(\int_{\Sigma_t} H_{k-1,F} F(\nabla u) F(\nu) d\mathcal{H}^{n-1}\right)^{k} \int_{\Sigma_t} H_{k-1,F} F(\nabla u)^k F(\nu) d\mathcal{H}^{n-1} \\
= \;& C_{n,k}\left\{[\zeta_{k-1}(\overline{\Omega_t})]^{n-k} \frac{d}{dt}\zeta_{k-1}(\overline{\Omega_t})\right\}^{k} \int_{\Sigma_t} H_{k-1,F} F(\nabla u)^k F(\nu) d\mathcal{H}^{n-1}.
\end{aligned}
$$

It follows

$$
\frac{[\zeta_{k-1}(\overline{\Omega_t})]^{n-k}}{\left[\frac{d}{dt}\zeta_{k-1}(\overline{\Omega_t})\right]^{k}} \leq C_{n,k} \int_{\Sigma_t} H_{k-1,F} F(\nabla u)^k F(\nu) d\mathcal{H}^{n-1}.
$$

Sketch of proof

By co-area formula,

$$
\begin{aligned}
I_{k,F}[u,\Omega] = \int_\Omega (-u) S_{k,F}[u] dx = \frac{1}{k}\int_\Omega S_k^{ij}[u] F F_i u_j \, dx \\
= \quad \frac{1}{k}\int_m^0 \int_{\Sigma_t} S_k^{ij}[u] F F_i u_j \frac{1}{|\nabla u|}\, d\mathcal{H}^{n-1} dt \\
= \quad \frac{1}{k}\binom{n-1}{k}\int_m^0 \int_{\Sigma_t} H_{k-1,F} F^k(\nabla u) F(\nu)\, d\mathcal{H}^{n-1} dt.
\end{aligned}
$$

It follows

$$
\begin{aligned}
I_{k,F}[u,\Omega] &= \frac{1}{k}\binom{n-1}{k}\int_m^0 \int_{\Sigma_t} H_{k-1,F} F^k(\nabla u) F(\nu)\, d\mathcal{H}^{n-1} dt \\
&\geq \kappa_n \binom{n}{k}\int_m^0 \frac{[\zeta_{k-1}(\overline{\Omega_t})]^{n-k}}{\left[\frac{d}{dt}\zeta_{k-1}(\overline{\Omega_t})\right]^k}\, dt \\
&= \kappa_n \binom{n}{k}\int_0^R r^{n-k}(\rho'_{k-1}(r))^{k+1}\, dr \\
&= I_{k,F}[u^*_{k-1}, \mathcal{W}_R].
\end{aligned}
$$

where $\rho_{k-1}(r) = u^*_{k-1,F}(x), r = F^0(x), R = \zeta_{k-1}(\bar{\Omega})$.

Thank you for your attention!