Projective manifolds whose tangent bundle is Ulrich

(with V. Benedetti, Y. Prieto, and S. Troncoso)

Pedro Montero

Universidad Técnica Federico Santa María
Valparaíso, Chile

Moduli, Motives and Bundles – New Trends in Algebraic Geometry (Oaxaca 2022)

- Motivation and preliminaries
- Constructing Ulrich bundles
- Results
- Some ingredients
- Sketch of proof

§1. Motivation and preliminaries

Classical question: Given $X = \{F = 0\} \subseteq \mathbf{P}^{n+1}(\mathbf{C})$ smooth hypersurface,

Can we find a matrix of linear forms (ℓ_{ij}) *such that* $F = \det(\ell_{ij})$*?*

Let $d = \deg(X)$. The first positive answers were:

- If $\dim(X) = 1$, this is **always** possible (**Dixon**, 1902).
- If $\dim(X) = 2$, this is always possible when $d = 1$ (linear algebra), $d = 2$ (since $X \cong \{x_0x_1 - x_2x_3 = 0\}$), and $d = 3$ (**Cayley**, 1869). However, we will see that for $d \geq 4$ the answers is **no**, in general.

⚠ If $\dim(X) \geq 3$, then $F = \det(\ell_{ij})$ defines a **singular** hypersurface. Thus, we rather consider the following question:

Given a smooth hypersurface as before, can we find $r \in \mathbf{N}^{\geq 1}$ *and a matrix of linear forms* (ℓ_{ij}) *such that* $F^r = \det(\ell_{ij})$*?*

Beauville (2000): Let $X = \{F = 0\} \subseteq \mathbf{P}^{n+1}(\mathbf{C})$ be a smooth hypersurface of $\deg(X) = d$. Then, for any $r \in \mathbf{N}^{\geq 1}$ we have

$F^r = \det(\ell_{ij}) \Leftrightarrow$ There is $E \to X$ vector bundle of rank r admitting $0 \to \mathcal{O}_{\mathbf{P}^{n+1}}(-1)^{\oplus rd} \xrightarrow{\ell} \mathcal{O}_{\mathbf{P}^{n+1}}^{\oplus rd} \to E \to 0$ linear resolution.

Theorem (Eisenbud-Schreyer-Weyman, 2003)

Let $X \hookrightarrow \mathbf{P}^N(\mathbf{C})$ be a smooth projective polarized n-fold, and $E \to X$ be a rank r vector bundle. The following are equivalent:

1. There is a linear resolution $0 \to \mathcal{O}_{\mathbf{P}^N}(-N+n)^{\oplus a_{N-n}} \to \ldots \to \mathcal{O}_{\mathbf{P}^N}(-1)^{\oplus a_1} \to \mathcal{O}_{\mathbf{P}^N}^{\oplus a_0} \to E \to 0.$
2. If $\pi : X \to \mathbf{P}^n$ is a finite linear projection, then $\pi_* E$ is trivial.
3. $\mathrm{H}^\bullet(X, E(-j)) = 0$ for every $j \in \{1, \ldots, n\}$.
4. $\mathrm{H}^i(X, E(-i)) = \mathrm{H}^j(X, E(-j-1)) = 0$ for every $i \geq 1$ and $j \leq n-1$.

In that case, we say that E is a **Ulrich bundle**.

Some interesting consequences:

Let $E \to X$ be an Ulrich bundle with respect to an embedding $X \hookrightarrow \mathbf{P}^N$ (i.e., with respect to a very ample divisor $H \subseteq X$). Then,

1. E is **arithmetically Cohen-Macaulay** (aCM) w.r.t. H, i.e.,

$$\mathrm{H}^i(X, E(jH)) = 0 \text{ for all } j \in \mathbf{Z} \text{ and } 0 < i < n.$$

   Moreover, $h^0(X,E) = \mathrm{rk}(E)\deg(X)$ where $\deg(X) = H^n \in \mathbf{N}^{\geq 1}$.

2. E is 0-regular (Castelnuovo-Mumford), thus **globally generated**.
3. If $Y \in |\mathcal{O}_X(1)|$ is a smooth hyperplane section, then $E|_Y$ is an Ulrich bundle w.r.t. $\mathcal{O}_X(1)|_Y$.

4. E is **slope semi-stable** with respect to H, i.e., for every non-zero subsheaf $\mathcal{F} \subseteq E$ we have $\mu_H(\mathcal{F}) \leq \mu_H(E)$, where

$$\mu_H(\mathcal{F}) \overset{\text{def}}{=} \frac{c_1(\mathcal{F}) \cdot H^{n-1}}{\mathrm{rk}(\mathcal{F})} \in \mathbf{Q}.$$

Conjecture (Bernd Ulrich, 1984)

Every smooth projective variety $X \hookrightarrow \mathbf{P}^N$ carry an Ulrich bundle.

⚠ Even in the (few) cases where the answer is known to be positive, it is interesting (and challenging) to determine the **Ulrich complexity**

$$\mathrm{uc}(X) := \min\{r \in \mathbf{N}^{\geq 1} \text{ s.t. there is a rank } r \text{ Ulrich bundle } E \to X\}.$$

Some (actually, many of the) known cases:

1. On $\mathbf{P}^n$, $E = \mathcal{O}_{\mathbf{P}^n}^{\oplus r}$ is a rank r Ulrich bundle.
2. On the quadric $\mathbf{Q}^n \subseteq \mathbf{P}^{n+1}$, the spinor bundles $\mathcal{S} \to \mathbf{Q}^n$ are Ulrich bundles (of rank $2^{[n-1/2]}$).
3. Let $X \subseteq \mathbf{P}^{n+1}$ be a smooth hypersurface of degree $d \geq 2$ with $\mathrm{Pic}(X) \cong \mathbf{Z}\mathcal{O}_X(1)$ (c.f. Noether-Lefschetz). Then, no line bundle $\mathcal{L} \cong \mathcal{O}_X(a)$ is Ulrich:

   Otherwise, $h^0(X, \mathcal{L}(-1)) = 0$ *and* $h^0(X, \mathcal{L}) = \mathrm{rk}(\mathcal{L})\deg(X) = d$ *would tell us that* $h^0(X, \mathcal{O}_X(a-1)) = 0$ *and* $h^0(X, \mathcal{O}_X(a)) \neq 0$*, and hence* $a = 0$*. This would imply that* $d = 1$*, which is impossible.*

   On the other hand, we have the following result:

(**Backelin-Herzog-Ulrich**, 1991): Every smooth complete intersection $X \subseteq \mathbf{P}^N$ admits an Ulrich bundle.

4. $\mathrm{Gr}(k,n)$ have equivariant Ulrich bundles (**Costa-Miró-Roig**, 2015). There are partial results for rational homogeneous spaces G/P.
5. (ESW, 2003): Every **curve** C admits an Ulrich line bundle, since it is enough to check the vanishing $h^0(C, E(-1)) = h^1(C, E(-1)) = 0$. If $\mathcal{L}$ is a general line bundle of degree $g-1$, then $E = \mathcal{L}(1)$ works[1].
6. Some minimal **surfaces**:
   - (a) $\kappa(S) = -\infty$ (Casanellas-Hartshorne, Miró-Roig-Pons-Llopins).
   - (b) $\kappa(S) = 0$ (Beauville, Aprodu-Farkas-Ortega, Faenzi).
   - (c) Some surfaces with $\kappa(S) = 1$ (Miró-Roig-Pons-Llopins).
   - (d) Some surfaces with $\kappa(S) = 2$ (Casnati, Lopez).
7. Some Fano threefolds with $\mathrm{Pic}(X) \cong \mathbf{Z}\mathcal{O}_X(1)$ (**Beauville**, 2017).

[1] This allows us to retrieve Dixon's result!

§2. Constructing Ulrich bundles

Besides commutative algebra methods, for surfaces we have:

1. Noether-Lefschetz type arguments (cf. **Aprodu-Farkas-Ortega**).
2. Cayley-Bacharach and Hartshorne-Serre construction (cf. **Beauville**).
3. Deformation theory arguments (cf. **Faenzi**).
4. Numerical characterization via Chern classes (cf. **Casnati**).

Cayley-Bacharach (CB) property

A finite subscheme Z of a smooth surface $S \hookrightarrow \mathbf{P}^N$ verify **CB** w.r.t. $\mathcal{O}_S(1)$ if: for every $C \in |\mathcal{O}_S(1)|$, the condition $Z \setminus \{\mathrm{pt}\} \subseteq C$ implies that $Z \subseteq C$.

Output: In that case, the Hartshorne-Serre construction give us

$$0 \longrightarrow \mathcal{O}_S(K_S) \longrightarrow E \longrightarrow \mathcal{I}_Z \otimes \mathcal{O}_S(1) \longrightarrow 0, \qquad (\star)$$

where E is a rank 2 **vector bundle** (!) and $\det(E) = \mathcal{O}_S(K_S + H)$.

Idea of the method:

Considering suitable $Z \subseteq S \subseteq \mathbf{P}^N$ (e.g. $N+2$ general points) we get $(\star)$. Tensoring by some convenient line bundle $\mathcal{L}$, we can produce an Ulrich bundle $E \otimes \mathcal{L}$ in many cases.

How to guess the right $\mathcal{L} \in \operatorname{Pic}(S)$?

(**Casnati**, 2017): Let $E \to S$ be a rank r vector bundle on the polarized surface $S \hookrightarrow \mathbf{P}\,\mathrm{H}^0(S, \mathcal{O}_S(H)) \cong \mathbf{P}^N$. Then, E is an Ulrich bundle iff

1. $\mathrm{H}^0(S, E(-H)) = \mathrm{H}^2(S, E(-2H)) = 0$.
2. $c_1(E) \cdot H = \frac{r}{2}(K_S + 3H) \cdot H$.
3. $c_2(E) = \frac{1}{2}(c_1^2(E) - c_1(E) \cdot K_S) - r(H^2 - \chi(S, \mathcal{O}_S)$.

Remark (**Lopez**, 2020): If $(X, \mathcal{O}_X(H))$ is a polarized n-fold ($n \geq 2$) with $\operatorname{Pic}(X) \cong \mathbf{Z}$, then for a rank r Ulrich bundle $E \to X$ we have

$$c_1(E) = \frac{r}{2}(K_X + (n+1)H).$$

§2. Constructing Ulrich bundles

Our starting point: Find a numerical characterization on 3-folds.

Slope Lemma (BMPT):

Let $(X, \mathcal{O}_X(H))$ be a polarized n-fold ($n \geq 2$), then for a rank r Ulrich bundle $E \to X$ we have

$$\frac{c_1(E) \cdot H^{n-1}}{r} \stackrel{\text{def}}{=} \mu_H(E) = \frac{1}{2}(K_X + (n+1)H) \cdot H^{n-1}.$$

For 3-folds we have the following (cf. Ciliberto-Flamini-Knutsen, 2022):

Proposition (BMPT):

Let $E \to X$ be a rank r vector bundle on the polarized 3-fold $X \hookrightarrow \mathbf{P}\,\mathrm{H}^0(X, \mathcal{O}_X(H)) \cong \mathbf{P}^N$. Then, E is an Ulrich bundle iff some identities "**à la Casnati**" hold (i.e., some cohomology groups have to vanish, and some identities involving $c_1(E) \cdot H^2$, $c_2(E) \cdot H$ and $c_3(E)$).

§3. Results

Key observation:

The Slope Lemma should be useful to study positivity of the tangent bundle (cf. Boucksom-Demailly-Păun-Peternell and Campana-Păun).

Natural question: In regard to the complexity of constructing Ulrich bundles, manifolds with canonically attached Ulrich bundles should be special. The starting point should be:

If T_X or Ω^1_X is an Ulrich bundle, what can we say about $X \hookrightarrow \mathbf{P}^N$?

Example (curves): Let $C \hookrightarrow \mathbf{P}^N$ be a degree $d = \deg(H)$ curve of genus g.

- If $\Omega^1_C \cong \mathcal{O}_C(K_C)$ Ulrich, $0 = h^1(K_C - H) = h^0(H) = N + 1 \Rightarrow\!\Leftarrow$
- If $T_C \cong \mathcal{O}_C(-K_C)$ Ulrich, then $0 = h^1(-K_C - H) = h^0(2K_C + H)$. The latter is $\neq 0$ if $g \geq 1$ by Riemann-Roch.
- If $C \cong \mathbf{P}^1$, we easily check that only $d = 3$ works (**twisted cubic**).

Example (surfaces): Let $S \hookrightarrow \mathbf{P}^N$ be a degree $d = H^2$ surface.

- If Ω^1_S Ulrich, the Slope Lemma implies that

$$c_1(\Omega^1_S) \cdot H \overset{\text{def}}{=} K_S \cdot H = 3H^2 + K_S \cdot H, \quad \text{i.e.,} \quad H^2 = 0 \;\Rightarrow\!\Leftarrow$$

Definition (ESW, 2003):

An Ulrich bundle $E \to X$ on the n-fold $(X, \mathcal{O}_X(H))$ is **Ulrich special**[a] if $\mathrm{rk}(E) = 2$ and $\det(E) \cong \mathcal{O}_X(K_X + (n+1)H)$.

[a](**Beauville**, 2000): $X = \{F = 0\} \subseteq \mathbf{P}^{n+1}$ with $F = \mathrm{Pf}(M)$ iff $\exists E \to X$ Ulrich special.

- If T_S Ulrich special, $c_1(T_S) \overset{\text{def}}{=} -K_S = K_S + 3H$ and hence $-2K_S = 3H$. In particular, S is a **del Pezzo surface** (i.e., $-K_S$ ample) and thus $\mathrm{Pic}(S) \cong \mathbf{Z}^\rho$ is torsion-free $\rightsquigarrow$ $-K_S = 3A$ for A ample.

  (**Kobayashi-Ochiai**): $S \cong \mathbf{P}^2$. In particular, we deduce that $\mathcal{O}_S(H) \cong \mathcal{O}_{\mathbf{P}^2}(2)$, i.e., $S \cong \mathbf{P}^2 \hookrightarrow \mathbf{P}^5$ (**Veronese surface**).

Main Theorem (Benedetti–M.–Prieto–Troncoso)

Let $X \hookrightarrow \mathbf{P}\,\mathrm{H}^0(X, \mathcal{O}_X(H)) \cong \mathbf{P}^N$ be a smooth projective n-fold. Then,

1. The cotangent bundle Ω^1_X is **never** Ulrich.
2. The tangent bundle T_X is Ulrich if and only if $(X, \mathcal{O}_X(H))$ is the **twisted cubic** $(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(3))$ or the **Veronese surface** $(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(2))$.

§4. Some ingredients

§4. Some ingredients

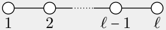
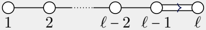
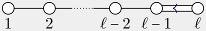
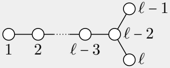
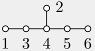
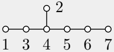
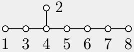
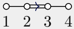
For surfaces, we can give a quick proof (not using Campana-Păun theorem) by means of Reider's solution to Fujita conjecture:

- (**Reider**, 1988): If D is a nef divisor on S s.t. $D^2 \geq 9$, then $K_S + D$ is very ample unless D satisfies some precise numerical restrictions.

For the general case, we need the following:

- $T_{\mathbf{P}^n}$ is never Ulrich if $n \geq 3$: $h^0(T_{\mathbf{P}^n}) = \dim \mathfrak{sl}_{n+1} > nH^n = nd^n$.
- If Ω^1_X or T_X is Ulrich, then the Slope Lemma implies that X is rationally connected (**Campana-Păun**, 2019) $\rightsquigarrow$ Ω^1_X is **not Ulrich**.
- If T_X is globally, then $X \cong A \times G/P$ is a homogeneous variety (**Borel-Remmert**, 1962) $\rightsquigarrow$ $X \cong G/P$ (since rationally connected).
- Slope Lemma: $d = H^n$ is a multiple of $n+2$ (resp. $\frac{n+2}{2}$) if n odd (resp. n even). In particular, $\dim \mathrm{Lie}(\mathrm{Aut}^\circ(X)) = h^0(X, T_X) \geq \frac{n(n+2)}{2}$.

§4. Some ingredients (G/P with $\mathrm{Pic}(G/P) \cong \mathbf{Z}$)

Lie algebra $\mathfrak{g}$	Dynkin diagram	$\dim_{\mathbb{C}} \mathfrak{g}$	$n = \dim_{\mathbb{C}}(G/P_r)$								
A_ℓ $(\ell \geq 1)$		$\ell^2 + 2\ell$	$r(\ell + 1 - r)$								
B_ℓ $(\ell \geq 2)$		$2\ell^2 + \ell$	$\frac{r}{2}(4\ell + 1 - 3r)$								
C_ℓ $(\ell \geq 3)$		$2\ell^2 + \ell$	$\frac{r}{2}(4\ell + 1 - 3r)$								
D_ℓ $(\ell \geq 4)$		$2\ell^2 - \ell$	$\frac{r}{2}(4\ell - 1 - 3r)$								
E_6		78	r	1	2	3	4	5	6		
			n	16	21	25	29	25	16		
E_7		133	r	1	2	3	4	5	6	7	
			n	33	42	47	53	50	42	27	
E_8		248	r	1	2	3	4	5	6	7	8
			n	78	92	98	106	104	97	83	57
F_4		52	r	1	2	3	4				
			n	15	20	20	15				
G_2		14	r	1	2						
			n	5	5						

§5. Sketch of Proof

Let $S \hookrightarrow \mathbf{P}\,\mathrm{H}^0(S, \mathcal{O}_S(H)) \cong \mathbf{P}^N$ be a surface with T_S Ulrich. Then,

1. Slope Lemma: $2K_S \cdot H = -3H^2 < 0$, i.e., K_S is not pseudo-effective, and thus $\kappa(S) = -\infty$. Actually, $S \sim_{\mathrm{bir}} \mathbf{P}^2$ (rationally connected).
2. A general curve $C \in |H|$ verifies
$$g(C) = 1 + \frac{1}{2}(H^2 + K_S \cdot H) = 1 - \frac{1}{4}H^2,$$
thus $\deg(S) = H^2 = 4$ and $K_S \cdot H = -6$.
3. Casnati's identities: $c_2(T_S) = \chi_{\mathrm{top}}(S) = K_S^2 - 8 + 2\chi(\mathcal{O}_S)$. Hence, Noether's identity gives $\chi(\mathcal{O}_S) = \frac{1}{5}(K_S^2 - 4)$.
4. Since $S \sim_{\mathrm{bir}} \mathbf{P}^2$, we have $1 = \chi(\mathcal{O}_S) = \frac{1}{5}(K_S^2 - 4)$ and thus $K_S^2 = 9$. It follows from the classification of minimal surfaces that $S \cong \mathbf{P}^2$. $\square$

§5. Sketch of Proof ($\dim(X) \geq 3$, $\rho(X) = 1$)

We know that if T_X is Ulrich, then $X \cong G/P$ is rational homogeneous. Assume that $\mathrm{Pic}(X) \cong \mathbf{Z}$. Then:

1. The Ulrich condition implies that $\dim \mathrm{Aut}^\circ(X) \geq \frac{n(n+2)}{2}$. Then, a case-by-case analysis shows that $X \cong \mathbf{P}^n$, $\mathbf{Q}^n \subseteq \mathbf{P}^{n+1}$ or $\mathrm{Gr}(2,5)$.
2. Actually, since $\deg(X)$ is a multiple of $n+2$ if n odd, we are reduced to analyse $\mathbf{Q}^{2m} \subseteq \mathbf{P}^{2m+1}$ and $\mathrm{Gr}(2,5)$.
3. For $X \cong \mathbf{Q}^{2m}$, we have $\deg(X) = (m+1)\ell$ for some $\ell \in \mathbf{N}^{\geq 1}$ and thus
$$2m(m+1)\ell = h^0(X, T_X) = \dim \mathfrak{so}_{2m+2} = (2m+1)(m+1) \;\Rightarrow\!\Leftarrow$$
4. For $X \cong \mathrm{Gr}(2,5)$, we have that
$$6\deg(X) = h^0(X, T_X) = \dim \mathfrak{sl}_5 = 24, \text{ i.e., } \deg(X) = 4.$$

This is impossible, as $\mathrm{Pic}(X) \cong \mathbf{Z}\mathcal{O}_X(1)$ with $\deg(\mathcal{O}_X(1)) = 5$. □

Final comments

1. For $X \cong G/P$ is such that $\rho(X) \geq 2$, the key remark is that $\mathrm{Pic}(G/P)$ is generated by homogeneous line bundles $\{L_i\}_{i \in \Sigma}$, and that $-K_X = j_i L_i$ with $j_i > 0$. Finally, we conclude from:
   - The Slope Lemma does not hold as long as each $j_i < \dim(X)$.
   - If there is $j_i \geq \dim(X)$ then $X \cong \mathbf{P}^n$, $\mathbf{Q}^n \subseteq \mathbf{P}^{n+1}$ or $\mathbf{P}^1 \times \mathbf{P}^{n-1}$.
2. Although we can exclude the case of abelian varieties here, it would be interesting to show the existence of Ulrich bundles on abelian 3-folds.
3. The existence of an Ulrich foliation $\mathcal{F} \subsetneq T_X$ should impose geometric restrictions on X. Also, what about twisted bundles $\Omega^1_X(k)$?
4. What can we say about the existence of Ulrich bundles on ball quotients $X \cong \mathbf{B}^n/\Gamma$?

Thanks for your attention!