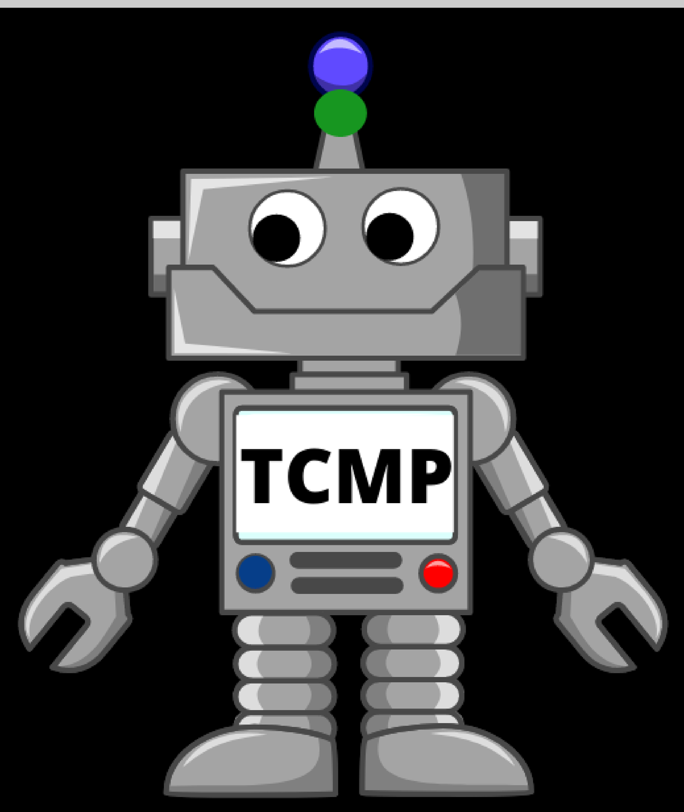
On the rational topological complexity of elliptic coformal spaces

Said Hamoun ⋆, Youssef Rami and Lucile Vandembroucq

Department of Mathematics, Faculty of Sciences, University of Moulay Ismail, Morocco

Centro de Matemática, University of Minho, Portugal

⋆Email: s.hamoun@edu.umi.ac.ma



TCMP: Topological complexity and motion planning

Objectives

Let X be a simply-connected space and let X_0 be its rationalization. We consider the rational topological comlexity and LS-category of X:

$$TC_0(X) := TC(X_0), \ cat_0(X) := cat(X_0).$$

The general goal is to study TC_0 for elliptic spaces. We say that X is (rationally) **elliptic** if

$$\dim \pi_*(X) \otimes \mathbb{Q} < \infty \quad \text{and} \quad \dim H^*(X;\mathbb{Q}) < \infty.$$

Many elliptic spaces (for instance homogenuous spaces G/H) admit a pure minimal Sullivan model $(\Lambda V, d)$ where **pure** means

$$dV^{even} = 0 \quad \text{and} \quad dV^{odd} \subset \Lambda V^{even}.$$

We will say that such a space is a pure elliptic space.

Sullivan models and TC_0

- The minimal Sullivan model $(\Lambda V, d)$ of X is a cochain algebra which is free as a commutative graded algebra and satisfies
$$d(V) \subset \Lambda^{\geq 2} V, \quad V \cong \pi_*(X) \otimes \mathbb{Q}, \quad H^*(\Lambda V, d) = H^*(X;\mathbb{Q}).$$
- X is formal if there is a quasi-isomorphism
$$(\Lambda V, d) \xrightarrow{\simeq} (H^*(X;\mathbb{Q}), 0)$$
- $TC_0(X) \leq n$ if and only if the projection
$$(\Lambda V \otimes \Lambda V, d) \to \left(\frac{\Lambda V \otimes \Lambda V}{(\ker \mu_{\Lambda V})^{n+1}}, \bar{d}\right),$$
where $\mu_{\Lambda V}$ is the multiplication of ΛV, admits a homotopy retraction [1].

Theorem

If X is a pure elliptic space which is formal then

$$TC_0(X) = 2cat_0(X) + \chi_\pi(X)$$

where $\chi_\pi(X) = \dim \pi_{even}(X) \otimes \mathbb{Q} - \dim \pi_{odd}(X) \otimes \mathbb{Q}$.

- Note that, for elliptic spaces, the homotopy characteristic satisfies
$$\chi_\pi(X) \leq 0.$$
- In the theorem above, if X is both formal and **coformal** ($dV \subset \Lambda^2 V$), then
$$TC_0(X) = \dim \pi_*(X) \otimes \mathbb{Q}$$
because in this case $cat_0(X) = \dim \pi_{odd}(X) \otimes \mathbb{Q}$ ([2]).

Theorem

If X is a pure elliptic coformal space, then we have

$$\dim \pi_{odd}(X) \otimes \mathbb{Q} + L_0(X) \leq TC_0(X) \leq \dim \pi_*(X) \otimes \mathbb{Q}$$

or, equivalently,

$$cat_0(X) + L_0(X) \leq TC_0(X) \leq 2cat_0(X) + \chi_\pi(X),$$

where $L_0(X)$ is a certain cuplength (see below).

Example

For the pure elliptic coformal space $X = \frac{SU(6)}{SU(3) \times SU(3)}$ we have

$$\dim \pi_{odd}(X) \otimes \mathbb{Q} = cat_0(X) = 3, \quad L_0(X) = 2,$$

and

$$TC_0(X) = 5 = \dim \pi_*(X) \otimes \mathbb{Q}.$$

About $L_0(X)$

For a pure coformal space the differential d is quadratic and splits in a some of

$$d_{p,q} : \Lambda^p V^{even} \otimes \Lambda^q V^{odd} \to \Lambda^{p+2} V^{even} \otimes \Lambda^{q-1} V^{odd}.$$

This induces a bigraduation $H^*(\Lambda V) = \bigoplus_{p,q} H^*_{p,q}(\Lambda V)$.

Roughly speaking $L_0(X)$ is given by

$$\max\{r : \exists \alpha_1 \cdots \alpha_r \neq 0, \ \alpha_i \in H^*_{odd,*}(\Lambda W)\}$$

where $(\Lambda W, d)$ is a certain extension of $(\Lambda V, d)$.

Additional results

We establish $TC_0(X) = \dim \pi_*(X) \otimes \mathbb{Q}$ for some special families of coformal pure elliptic spaces e.g.

$$\Lambda(x_1, \cdots, x_n, y_1, \cdots, y_n, u) \quad dx_i = 0, \, dy_i = x_i^2, \, du = \sum \alpha_{ij} x_i x_j$$

where $deg(x_i)$ is even and $\alpha_{ij} \in \mathbb{Q}$.

This includes spaces for which $cat_0(X) + L_0(X) < TC_0(X)$.

Work in progress/ Future work

- Investigate on the equality $TC_0(X) = \dim \pi_*(X) \otimes \mathbb{Q}$ for elliptic coformal spaces and, more generally, on the equality $TC_0(X) = 2cat_0(X) + \chi_\pi(X)$.
- Study the non-pure case.

Note that for, any elliptic space X where $\pi_*(X) \otimes \mathbb{Q}$ is concentrated in odd degrees, we have [3]

$$TC_0(X) = cat_0(X) = \dim \pi_{odd}(X) \otimes \mathbb{Q} = 2cat_0(X) + \chi_\pi(X).$$

References

[1] Carrasquel-Vera, José Gabriel *The rational sectional category of certain maps.* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 17 (2017).

[2] Félix, Yves; Halperin, Stephen *Rational LS category and its applications.* Trans. Amer. Math. Soc. 273 (1982).

[3] Jessup, Barry; Murillo, Aniceto; Parent, Paul-Eugène *Rational topological complexity.* Algebr. Geom. Topol. 12 (2012).

Acknowledgment

S.H would like to acknowledge the CNRST-Morocco, the laboratory of Pure Mathematics, Faculty of Sciences, University of Moulay Ismail for support and CMO-Oaxaca México for hospitality.