On the sectional category of subgroups inclusions and Adamson cohomology theory

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Sectional category of subgroup inclusions

The inclusion $i\colon H \hookrightarrow G$ induces a map $K(i,1)\colon K(H,1) \to K(G,1)$ between the corresponding classifying spaces, satisfying $\pi_1\big(K(i,1)\big) = i$. Define the *sectional category* of i, denoted $\mathrm{secat}(i)$, as the sectional category of $K(i,1)$.
It is possible to characterize $\mathrm{secat}(i)$ as following

Lemma. *Let X be the universal covering space of a $K(G,1)$ space. Then $\mathrm{secat}(H \hookrightarrow G) \leq n$ if and only if*

$$p_n\colon X \times_G *^{n+1}(G/H) \to X/G$$

has a section.

And, from that, obtain a generalization in spirit of [2]

Proposition. *Let X be the universal covering space of a $K(G,1)$ space. The sectional category of $H \hookrightarrow G$ coincides with the smallest integer $n \geq 0$ such that there exists a G-equivariant map*

$$X \to *^{n+1}(G/H).$$

Adamson relative cohomology

The Adamson relative cohomology depends on a relative homological algebra theory, developed by G. Hochschild in [3]. The key notions are relative exactness and projectiveness.

Definition. *An exact sequence of G-modules $\cdots \to M_i \xrightarrow{f_{i+1}} M_i \xrightarrow{f_i} M_{i-1} \to \cdots$ is said to be (G,H)-exact provided that $\ker f_i$ is a direct summand of M_i as H-module for each i.*
A G-module P is said to be (G,H)-projective provided that for every short (G,H)-exact sequence of modules $M \xrightarrow{f} N \to 0$ and every G-homomophism $g\colon P \to N$, there exists a G-homomorphism $h\colon P \to M$ such that $fh = g$.

As a (G,H)-projective resolution of $\mathbb{Z}$, we have the *relative standard resolution*, with $C_*(G/H) = \mathbb{Z}[(G/H)^*]$.

Definition. *The relative extension functor is defined as $\mathrm{Ext}^n_{(G,H)}(M,N) = H^n(\mathrm{Hom}_G(P_M,N))$ with M and N G-modules, and P_M a (G,H)-projective resolution of M. Define the Adamson relative cohomology of G with respect to H and coefficientes M as*

$$H^*([G:H],M) = \mathrm{Ext}^*_{(G,H)}(\mathbb{Z},M).$$

We will denote the Adamson cohomological dimension of G relative to H *as* $cd[G:H]$.

Adamson vs Bredon cohomology

Relative canonical class and universality

Consider a permutation G-module $\mathbb{Z}[G/H]$ and I the kernel of the augmentation $\mathbb{Z}[G/H] \to \mathbb{Z}$. Define a G-module homomorphism $\xi\colon \mathbb{Z}[G] \otimes K \to I$ as the composition of $\varepsilon \otimes \mathrm{id}$ and the map $\mu\colon K \to I$ induced by the canonical projection $G \to G/H$. This is a cocycle, and thus it represents a one-dimensional cohomology class.

Definition. *The Bernstein class of G relative to H is the class $\omega \in H^1(G,I)$ represented by ξ.*

The n-th power $\omega^n \in H^n(G, I^{\otimes n})$ can be seen as represented by the map

$$\mathbb{Z}[G] \otimes K^{\otimes n} \xrightarrow{\varepsilon\otimes\mathrm{id}} K^{\otimes n} \xrightarrow{\mu^{\otimes n}} I^{\otimes n}.$$

For the diagonal inclusion, this coincides with the class in [1]. An analogue can be defined also in the Adamson cohomology context. For that, consider the augmentation short exact sequence:

$$0 \to I \hookrightarrow \mathbb{Z}[G/H] \xrightarrow{\varepsilon} \mathbb{Z} \to 0.$$

If we tensor by $I^{\otimes k}$, we get a $(G:H)$-projective resolution of the form

$$\cdots \to \mathbb{Z}[G/H] \otimes I^{\otimes k} \to \mathbb{Z}[G/H] \otimes I^{\otimes k-1} \to \cdots \to \mathbb{Z}[G/H] \to \mathbb{Z} \to 0.$$

So given $\alpha \in H^n([G:H],M)$ we have $\alpha = [a]$ for some $a : \mathbb{Z}[G:H] \otimes I^{\otimes n} \to M$.

Definition. *Define the* cup product $[a][b] \in H^{p+q}([G:H], A \otimes B)$ *as the class represented by the map*

$$\mathbb{Z}[G/H] \otimes I^{\otimes p+q} \xrightarrow{\varepsilon\otimes\mathrm{id}} I^{\otimes p+q} \xrightarrow{\hat{a}\otimes\hat{b}} A \otimes B.$$

Definition. *We define the* relative canonical class, *or* Adamson canonical class, *$\phi \in H^1([G:H],I)$, as the class represented by the cocycle*

$$\mathbb{Z}[G/H] \otimes I \xrightarrow{\varepsilon\otimes\mathrm{id}} I.$$

The canonical class defined is universal, in the following sense

Proposition. *For any G-module M, and any cohomology class $\lambda \in H^n([G:H],M)$ there exists $\psi\colon I^{\otimes n} \to M$ such that $\psi^*(\phi^n) = \lambda$.*

Definition. *Define* $\mathrm{height}(\phi)$ *as the largest $n \geq 0$ such that $\phi^n \in H^n([G:H], I^{\otimes n})$ is nonzero.*

The Adamson cohomology groups can be characterised in terms of zero divisors for suitable coefficient systems.

Proposition. *For any G-module M and $n \geq 1$, we have*

$$H^n([G:H],M) = \ker\left[H^1(G, \mathrm{Hom}_{\mathbb{Z}}(I^{\otimes n-1},M)) \to H^1(H, \mathrm{Hom}_{\mathbb{Z}}(I^{\otimes n-1},M))\right].$$

In particular,

$$H^1([G:H],M) = \ker\left[H^1(G,M) \to H^1(H,M)\right].$$

It is possible to obtain a generalization of the theorem of Costa and Farber of [1] relating $\mathrm{TC}(X)$ and the height of the Bernstein class of G relative to H

Theorem. *Let $H \leq G$ be discrete groups, with $n = cd(G) \geq 3$ and $i\colon H \to G$ the inclusion as a subgroup. Then $\mathrm{secat}(i) \leq n-1$ if and only if $\omega^n = 0$.*

This can be proved solving an obstruction theoretical problem which relates both relative canonical classes.

The universal space of a family of subgroups

For the Bredon cohomology, it is necessary to consider structure relative to a family of subgroups. Recall that a family $\mathcal{F}$ of subgroups of G is said to be *full* provided that it is non-empty, closed under conjugation and taking subgroups, and the condition $H, K \in \mathcal{F}$ implies that $H \cap K \in \mathcal{F}$. We will write $\langle H \rangle$ for the smallest full family of subgroups of G containing H.

Definition. *Define the classifying G-CW complex with respect to $\mathcal{F}$, $E_{\mathcal{F}}(G)$, as a G-CW complex satisfying:*

- *Every isotropy subgroup of an element of $E_{\mathcal{F}}(G)$ belongs to $\mathcal{F}$.*
- *$E_{\mathcal{F}}(G)$ is a terminal object in the subcategory of G-CW complexes satisfying the above condition, i.e, for every other G-CW complex X with all its isotropy subgroups in $\mathcal{F}$, there exists, up to homotopy, one G-equivariant map from X to $E_{\mathcal{F}}(G)$. $E_{\mathcal{F}}(G)$ is unique up to G-homotopy.*

The projective resolution for defining Bredon cohomology comes from a cellular chain complex. Recall that $\mathcal{O}_{\mathcal{F}}(G)$ denotes the orbit category of G with respect to $\mathcal{F}$.

Definition. *Let G be a group, $\mathcal{F}$ a full family of subgroups, and X a G-CW complex with isotropy subgroups in $\mathcal{F}$. We define the cellular $\mathbb{Z}\mathcal{O}_{\mathcal{F}}(G)$-chain complex as a contravariant functor*

$$\begin{aligned} C_*^{\mathcal{F}}(X)\colon\ \mathcal{O}_{\mathcal{F}}(G) &\longrightarrow Ch_*(Ab) \\ G/H &\longmapsto C_*(X^H) \end{aligned}$$

Recall that a Bredon module with respect to $\mathcal{F}$ is defined as a diagram from the category $\mathcal{O}_{\mathcal{F}}(G)$ to the category of abelian groups. The Bredon cohomology of a G-CW complex can be defined as follows

Definition. *With the notation above, we define the Bredon cohomology of X with coefficients in $\underline{M}$ as*

$$H^*_{\mathcal{F}}(X,\underline{M}) = H^*(\mathrm{Hom}_{\mathbb{Z}\mathcal{O}_{\mathcal{F}}(G)}(C_*^{\mathcal{F}}(X),\underline{M}))$$

We will denote the Bredon cohomological dimension of G with respect to $\mathcal{F}$ as $cd_{\mathcal{F}}(G)$.

Comparison between Adamson and Bredon cohomology

For comparing Adamson and Bredon cohomologies, evaluate the cellular $\mathbb{Z}\mathcal{O}_{\langle H\rangle}(G)$-chain complex on the principal component, with gives us $C_n^{\langle H\rangle}(E_{\langle H\rangle}(G))(G/e) = H_n(E_{\langle H\rangle}(G)_{n+1}, E_{\langle H\rangle}(G)_n)$. It can be seen that $C_n^{\langle H\rangle}(E_{\langle H\rangle}(G))(G/e) = \mathbb{Z}[(G/H)^{n+1}]$.

For every G-module M take the associated Bredon module $\underline{M}$ taking values $\underline{M}(G/H) = M^H$, with $\underline{M}(f)$ being the action of g on M for the g associated to the equivariant map $f\colon G/H \to G/K$.

Define now, for every $n \geq 0$ the map

$$\Phi\colon \mathrm{Hom}_G(\mathbb{Z}[(G/H)^n],M) \to \mathrm{Hom}_{\mathbb{Z}\mathcal{O}_{\langle H\rangle}(G)}(C_n^{\langle H\rangle}(E_{\langle H\rangle}(G)),\underline{M})$$

by assigning to every $\varphi \in \mathrm{Hom}_G(\mathbb{Z}[(G/H)^n],M)$ a map φ_K for every subgroup $K \in \langle H\rangle$, defined as the composition

$$\mathbb{Z}[((G/H)^K)^n] \hookrightarrow \mathbb{Z}[(G/H)^n] \xrightarrow{\varphi} M$$

where the first map is the inclusion. With this, we can compare Bredon with Adamson cohomology.

Theorem. *Let $H \leq G$ be subgroups, M a G-module, and $\underline{M}$ the Bredon module satisfying $\underline{M}(G/K) = M^K$. Then*

$$H^*([G:H],M) \cong H^*_{\langle H\rangle}(G,\underline{M}).$$

And as a consequence, $cd[G:H] \leq cd_{\langle H\rangle}(G)$.

So Adamson is a candidate for a better upper bound for both $\mathrm{TC}(X)$ and $\mathrm{secat}(i)$ than the Bredon one.

Final remarks for $\mathrm{secat}(i)$

It is possible to define a cohomology class represented by the short exact sequence of Bredon modules

$$0 \to \underline{I} \to \mathbb{Z}[?,G/H] \xrightarrow{\varepsilon} \underline{\mathbb{Z}} \to 0.$$

This class is the canonical class associated to the family $\langle H\rangle$. From [2] can be seen that it is universal. Its image in cohomology via the induced map of a particular equivariant map $\rho\colon E(G) \to E_{\langle H\rangle}(G)$ in the principal component coincides with the Bernstein relative class.

Proposition. *Given $i\colon H \hookrightarrow G$ a subgroup inclusion, and n the greatest integer such that*

$$H^n_{\langle H\rangle}(G,\underline{M}) \xrightarrow{\rho^*} H^n(G,M)$$

is nonzero for some M, then $n \leq height(\omega) \leq \mathrm{secat}(i)$

This shows that one of the conjectures of [2] is, in the context of topological complexity, another take on the classical conjecture relating $TC(X)$ and the height of the Costa-Farber class.

In general, $\mathrm{secat}(i)$ cannot be equal to Adamson or Bredon cohomological dimension. As an example, consider the short exact sequence of groups

$$1 \to [F_n,F_n] \to F_n \to \mathbb{Z}^n \to 0$$

with F_n the free group of n generators, and the inclusion of the commutator subgroup. Again $\mathrm{height}(\phi) \geq cd(\mathbb{Z}^k)$. The examples relies strongly on selecting normal subgroups, so this does not give any indication that the conjecture is not true in the setting of topological complexity.

References

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Acknowledgements

Funded by 2016/21/P/ST1/03460 Polonez 2 project "Algebraic topology and its applications" co-funded with 944 874PLN (220 039EUR) by the Polish National Science Center and European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 665778.