Geometric and Variational Methods in Celestial Mechanics

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1 Overview of the Field

Determining qualitative properties of the long term behavior of the planets of the Solar system, and now also of extra Solar systems, is the oldest question in Dynamical Systems. The simplest model of the Solar system is the N-body problem: N bodies moving under the influence of the Newtonian gravitational force. Even if it has been studied from more than two centuries and despite the outstanding recent progress, the N-body problem is far from being well understood.

The goal of this conference was to bring together different mathematicians who study Celestial Mechanics models from different perspectives and with different techniques to achieve a deeper understanding of the dynamics of the N-body problem.

2 Presentation Highlights

Tere Martinez-Seara, Universitat Politècnica de Catalunya
Chaos in the three body problem.
Monday, June 20, 9:00-9:40

Professor Martinez-Seara is a well established reseracher. She opened the workshop and presented several results about chaos in the solar system. She considered the planar three body problem with masses $m_0, m_1, m_2 > 0$, and assume that all three are not equal. We consider a Poincaré map defined on a section of the phase space. After reduction of the problem by its first integrals this is a 4-dimensional map. Professor Seara constructed a hyperbolic invariant set of the Poincaré map where its dynamics is conjugated to the (infinite symbols) Bernouilli shift. As a consequence we prove the existence of chaotic motions and positive topological entropy for the planar three body problem. The chaotic behaviour also provides the existence of interesting motions like the oscillatory ones for the planar three body problem. This is a joint work with M. Guardia, P. Martin and J. Paradela

Irene De Blasi, University of Turin
An example of billiard in Celestial Mechanics
Monday, June 20, 9:40-10:20

Irene De Blasi is a younger researcher. She presented a new type of dynamical model, describing the motion of a point-mass particle in an elliptic galaxy with a massive central core (such as, for example, a Black Hole), is studied. This kind of model belongs to the more general class of the refraction billiards, which are particularly useful as a way to describe the dynamics of particles under the action of discontinuous potentials. In our case, a refraction interface (a regular closed curve) separates a Keplerian potential with positive energy from a two-dimensional homogeneous harmonic potential.

The dynamical properties of the system depend crucially on the geometric fea- tures of the interface. In particular, this talk focuses on three main aspects: the linear stability of particular (homothetic) equilibrium trajectories, the existence of orbits with prescribed rotation number for close-to-circle interfaces and the existence of a symbolic dynamics under more general assumptions. Work in collaboration with V. Barutello and S. Terracini.

Donato Scarcella, Université Paris-Dauphine
Asymptotic quasiperiodic solutions for time dependent Hamiltonians.
Monday, June 20, 10:20-11:00

In 1954 Kolmogorov, with a surprising work, laid the foundation for the so-called KAM theory. This result was followed by the ones of Arnold, Moser and many other mathematicians. The KAM theory shows the persistence of quasiperiodic solutions (that is to say a superposition of finitely many oscillatory motions of different frequencies), in nearly integrable Hamiltonian systems. It is motivated by classical problems in celestial mechanics such as the n-body problem

In this talk Donato Scarcella was interested in perturbations which depend on time. He analyzed some properties associated to suitable time dependent Hamiltonians converging asymptotically in time to autonomous Hamiltonians having an invariant torus supporting quasiperiodic solutions.

Donato Scarcella studied the conditions of existence of solutions which converge asymptotically in time to the quasiperiodic solution of the unperturbed autonomous system. Moreover, we will analyze the exemple in celestial mechanics of a planetary system perturbed by a given comet coming from and going back to infinity, asymptotically along a hyperbolic Keplerian orbit.

Ezequiel Maderna, Universidad de la República (Uruguay)
Buseman functions and hyperbolic motions in the newtonian N-body problem. Part 1
Monday, June 20, 11:30-12:10

Given a non-collision configuration a on the N-body problem, we construct a special viscosity solution b of the Hamilton-Jacobi equation so that all calibrated curves of b are hyperbolic motions (as defined by J.Chazy) with limit shape given by a. Moreover, we show that such a viscosity solution is unique up to an additive constant. in this talk Prof. Maderna showed that this results implies that for any initial configuration x_0 there is a hyperbolic motion x starting at x_0 at time $t = 0$ and with limit shape given by a.

Lei Zhao, University of Augsburg
$z \to z^2$
Monday, June 20, 12:10-12:50

This simple complex analytic mapping in the title induces a well-known correspondence between the planar Hooke and Kepler problems, and regularizes the planar Kepler problems with additional regular perturbations. The mapping has several extensions and many applications. In this talk Lei Zhao illustrated its role yet more by presenting some recent results concerning enumerations of generalized periodic orbits in a periodically-forced Kepler problem, the construction of several new types of integrable mechanical billiards, its relation to a forgotten theorem of Darboux and the Bertrand systems on abstract surface of revolutions. The talk is based on several joint works with A.Boscaggin-R.Ortega, with A. Takeuchi, and with A. Albouy.

Ke Zhang, University of Toronto
Gevrey growth for the formally linearizable billiard, after Treschev.
Monday, June 20, 12:50-13:30

Treschev made the remarkable discovery that there exists formal power series describing a billiard with locally linearizable dynamics. We show that if the frequency for the linear dynamics is Diophanine, the Treschev example is $(1 + \alpha)$-Gevrey for some $\alpha > 0$. Professor Zhang's proof was based on a KAM-like iterative scheme. Hopefully, our result sheds a light on the more important question of whether this example is convergent. This is a joint work with Qun Wang (University of Toronto).

Guowei Yu, Chern Institute of Mathematics, Nankai University
Global Surfaces of Section and Periodic Orbits in The Spatial Isosceles Three Body Problem.
Tuesday, June 21st, 9:00-9:40

In this talk, Professor Yu considered the spatial isosceles three body problem. For certain choices of energy and angular momentum, we find some disk-like global surfaces of section with the Euler orbit as their common boundary, and a brake orbit passing through them. By considering the Poincare maps of these global surfaces of section, we show generically there are infinitely many periodic orbits. Moreover under certain assumption, we find there are infinitely many different kinds of brake and symmetric periodic orbits. This is a joint work with Xijun Hu, Lei Liu and Yuwei Ou.

Mar Giralt, Universitat Politècnica de Catalunya
Chaotic co-orbital motions to L^3 in the Restricted Planar Circular 3-Body Problem.
Tuesday, June 21st, 9:40-10:20

Mar Giralt considered the Restricted Planar Circular 3-Body Problem (RPC3BP) with ratio between the masses of the primaries small. This configuration can be modeled by a two degrees of freedom autonomous Hamiltonian. It has a saddle-center equilibrium point called L3 (collinear with the primaries and beyond the largest one) with a 1-dimensional stable and unstable manifold. Moreover, the modulus of the hyperbolic eigenvalues are weaker than the elliptic ones. In this work, we prove the existence of Lyapunov periodic orbits exponentially close to L3 with stable and unstable invariant manifolds that intersect transversally. Then, by the Smale-Birkhoff homoclinic theorem, this implies the existence of chaotic motions on a neighborhood of L3 and its invariant manifolds. To prove this result, we present an asymptotic formula for the distance between the stable and unstable manifolds of L3. Due to the rapidly rotating dynamics, this distance is exponentially small with respect to the mass ratio and, as a result, classical perturbative methods (i.e the Melnikov-Poincaré method) can not be applied. This is a joint work with Inma Baldomá and Marcel Guardia.

Rafael de la Llave, Georgia Institute of Technology
Quasiperiodic solutions in some dissipative systems: Theory and computation.
Tuesday, June 21st, 10:20-11:00

Professor de la Llave considered conformally symplectic systems. These are systems that transform a symplectic form into a multiple of itself. Some practical examples are: mechanical systems with friction proportional to the velocity, Euler Lagrange equations for exponentially discounted systems (optimizing the present value of operations in the presence of inflation).

A method to obtain proofs of existence of quasi-periodic solutions is to device a Newton method for an invariance equation which takes advantage of geometric identities.

This method, besides proofs leads to very efficient (low operation count per step, low storage requirements but still quadratic convergence) algorithms. Note that the method leads to results in a-posteriori form (an approximate solution of the invariance equation with good condition numbers implies the existence of a true solution close to the approximate one).

On the theoretical side, we present results on Lagrangian tori as well as on whiskered tori. One advantage of the method with respect to more traditional methods based on normal forms is that it can study whiskered tori when the stable and unstable bundle are non-trivial.

Besides the theoretical results, we will present numerical results which show computations close to the breakdown. Joint work with R. Calleja, A. Celletti and J. Gimeno.

Vivina Barutello, University of Torino
Regularized variational principles for the perturbed Kepler problem.
Tuesday, June 21st, 11:30-12:10

Professor Barrutello proposed a new method that combines the use of variational techniques with regularization methods in order to study existence and multiplicity results for the periodic and the Dirichlet problem associated to the perturbed Kepler system. This is a joint research with R. Ortega and G. Verzini.

Andrea Venturelli, Avignon Université
Buseman functions and hyperbolic motions in the newtonian N-body problem. Part 2
Tuesday, June 21st, 12:10-12:50

Given a non-collision configuration a on the N-body problem, we construct a special viscosity solution b of the Hamilton-Jacobi equation so that all calibrated curves of b are hyperbolic motions (as defined by J.Chazy) with limit shape given by a. Moreover, we show that such a viscosity solution is unique up to an additive constant. This results implies that for any initial configuration x_0 there is a hyperbolic motion x starting at x_0 at time $t = 0$ and with limit shape given by a. Moreover, in this talk Prof. Venturelli showed that this hyperbolic motion is unique for almost every initial configuration x_0.

Slim Ibrahim, University of Victoria
Dynamical classification of the two-body and Hill's lunar problems with quasi-homogeneous potentials.
Tuesday, June 21st, 12:50-13:30

As studied in many examples, higher order correction added to the Newtonian potential often provides more realistic and accurate quasi-homogeneous models in astrophysics. Important examples include the Schwarzschild and the Manev potentials.

The quasi-homogeneous N-body problem aims to study the interaction between N point particles under a prescribed potential. The classical (Newtonian) Hill's lunar problem aims to improve the solution accuracy of the lunar motion obtained by solving the two-body (Earth-Moon) system. Hill's lunar equation under the Newtonian or homogeneous potentials has been derived from the Hamiltonian of the three-body problem in a uniform rotating coordinate system with angular speed ω, by using symplectic scaling and heuristic arguments on various physical quantities.

In this talk, Professor Ibrahim first introduced a new variational method characterizing relative equilibria with minimal energy. This enables us to classify the dynamic in terms of global existence and singularity for all possible ranges of the parameters.

Then he derived Hill's lunar problem with quasi-homogenous potential, and finally, we implement the same ideas to demonstrate the existence of "black hole effect" for a certain range of the parameters: below and at some energy threshold, invariant sets (in the phase space) with non-zero Lebesgue measure that either contain global solutions or solutions with singularity are constructed.

Alberto Boscaggin, Università degli studi di Torino
Periodic solutions to relativistic Kepler problems.
Wednesday, June 22nd, 9:00-9:40

By using both geometrical and variational methods, Professor Boscaggin presented the existence, multiplicity and variational characterization of periodic solutions of (forced) relativistic Kepler problems in the plane. Joint works with W. Dambrosio (Torino), G. Feltrin and D. Papini (Udine)

Gian Marco Canneori, Universitá di Torino
Symbolic dynamics for the anisotropic N-centre problem.
Wednesday, June 22nd, 9:40-10:20

Professor Canneori considered the planar N-centre problem of Celestial Mechanics in which we associate to every centre a homogeneous anisotropic potential. As a peculiar difference with classical Newtonian potentials, the attraction field generated by every centre is no longer radial and the rotational symmetry is lost. As a consequence, the integrability of the system is completely lost and regularization methods such as the Levi-Civita transform no longer succeed in this contest. The total energy is still a first integral of the system and our main result is the existence of bounded periodic trajectories in negative energy shells. This is obtained employing both variational and perturbation methods. In particular, we show that over a certain threshold on the homogeneity degree of the potentials all the solution trajectories of this system are collision-less. Consequently, the existence of a symbolic dynamics for the system is deduced, which is enriched by the important role played by the minimal central configurations of the potentials taken into account. This is a joint work with Vivina Barutello and Susanna Terracini.

Gabriella Pinzari, Universitá di Padova
Recent results on the three—body problem.
Wednesday, June 22nd, 10:20-11:00

The planar Euler problem has a first integral which, in the limit when the mass of one of the attracting centers vanishes, has a pendulum-like dynamics. On the other hand, the three—body problem Hamiltonian differs from the one of the Euler problem for a kinetic term. Therefore, the natural question arises whether, in the three—body problem there is a memory of the dynamics of the Euler problem. Professor Pinzari showed that, under suitable assumptions, this is the case. This talk is based on joint works with Qinbo Chen, Jerome Daquin and Sara Di Ruzza.

Ariadna Farres, NASA's Goddard Space Flight Center
On the geometry of Station-Keeping around Libration Point Orbits.
Wednesday, June 22nd, 11:30-12:10

Over the years Libration Point orbits have become relevant in space applications, as the regions around the L1 and L2 points in the Circular Restricted Three-Body (CRTBP) are ideal for space weather missions like the Space Weather Follow-On (SWFO) or deep space observations like the Roman Space Telescope (RST) or the James Webb Space Telescope (JWST). It is well known that Libration point orbits are unstable and require routine station-keeping maneuvers to remain in orbit. Over the years different station-keeping strategies have been proposed, each of them with different objective functions. In this talk, Professor Farres described the geometry behind some of these strategies making use of the Floquet mode reference frame to compare them. We focus on two different strategies, the widely used by NASA approach, x-axis velocity constraint at the plain crossing, and the Floquet mode approach. The first approach looks for the delta-v required to ensure that at the 4th plane crossing Vx = 0, ensuring the spacecraft to orbit around the Libration point. On the other hand, the Floquet mode approach looks for the delta-v that cancels the unstable, bringing the trajectory close to the stable manifold or a reference orbit. As we will see, both approaches have some basic geometrical common features, even when they are compared using high-order techniques. Understanding the geometry of these strategies allows us to study the optimality of the delta-v maneuvers analytically. We will show why the minimum thrust direction is close to the position components of the stable eigenvector. We also describe the dependence of the maneuver size as a function of the thrust direction and provide a tool to describe the impact of mission constraints in the thrust directions. Moreover, the geometric interpretation of station-keeping allows us to envision strategies that can help to mitigate the impact of mission constraints, such as actively biasing the delta-v maneuvers when the spacecraft can only thrust away from the sun. This is a joint work with Chen Gao, Josep Masdemont, Gerard Gomez, Dave Folta and Cassandra Webster (Geometrical Analysis of Station-Keeping Strategies About Libration Point Orbits, JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS Vol. 45, No. 6, June 2022)

Jaime Paradela, Universitat Politecnica de Catalunya
Arnold Diffusion in the Restricted Planar Three Body Problem.
Wednesday, June 22nd, 12:10-12:50

Consider the RPE3BP with any mass ratio $\mu \in (0, 1/2)$ and any eccentricity $\in (0, 1)$. Jaime Paradela builded orbits along which the angular momentum of the massless body becomes unbounded as $t \to \infty$.

The construction relies on an Arnold diffusion mechanism by designing a transition chain of periodic orbits. For that, we identify an invariant manifold at infinity and prove that its four dimensional stable and unstable manifolds intersect transversally along two different homoclinic manifolds. These homoclinic manifolds define two scattering maps which encode the dynamics along the heteroclinic orbits. We prove that they can be combined to build a sequence of periodic orbits connected by heteroclinics along which the angular momentum grows unboundedly. One of the main diculties is that the splitting angle between the stable and unstable manifolds is exponentially small with respect to the angular momentum and therefore Melnikov theory cannot be applied. This is joint work with Marcel Guardia and Tere Seara.

Joan Gimeno, Georgia Tech/UB
Computation of invariant manifolds of high dimensional tori in Celestial Mechanics.
Wednesday, June 22nd, 12:50-13:30

In this talk, Dr. Gimeno presented a procedure to compute reducible tori and their un/stable manifolds. The method is efficient and parallelizable and it scales to multiple shooting approaches. To illustrate the results we are going to focus in a perturbed model of the Earth-Moon system with five natural frequencies leading to high dimensional quasi-periodic solutions of the perturbed R3BP. This was a joint work with A. Jorba, B. Nicolas, and E. Olmedo.

Kuo-Chang Chen, National Tsing Hua University
Some quantitative aspects of action minimizing solutions.
Thursday, June 23rd, 9:00-9:40
Minimizing methods have been successfully applied to construct various types of periodic solutions for the n-body and n-center problems during the past two decades. Majority of relevant researches were endeavored to understand qualitative features such as existence, uniqueness, and stability. In this talk we discuss a topic with relatively less attention — quantitative estimates for action values and mutual distances for action minimizing solutions. Professor Chen demonstrated some simple but nontrivial bounds. These estimates will facilitate numerical explorations to effectively locate and search new orbits.

Vadim Kaloshin, University of Maryland
On non co-preservation of 2- and 3-rational caustics for nearly circular billiards.
Thursday, June 23rd, 9:40-10:20

In this talk, Professor Kaloshin discussed recent results concerning the question of co-preservation of two rational caustics by nearly circular billiard tables. In particular, he gave a partial answer to a conjecture by Tabashnikov. This is based on a joint work with Edmond Koudjinan and Ke Zhang.

Carlos García Azpeitia, IIMAS-UNAM
Braids of the N-body problem by cabling multiple central configurations.
Thursday, June 23rd, 10:20-11:00

Professor García Azpeitia proved the existence of relative periodic solutions of the planar $N = \sum_{j=1}^{n} k_j$-body problem starting with n bodies moving close to a non-degenerate central configuration and replacing each of them with clusters of k_j bodies that move close to a small central configuration. We name these solutions carousel solutions. The proof relies on blow-up techniques for variational methods.

Qun Wang, Henan University
The n-vortex problem on a sphere.
Thursday, June 23rd, 11:30-12:10

Professor Wang investigated the dynamical behaviours of the n-vortex problem with arbitrary circulations on a sphere, equipped with an arbitrary metric g. By mixing information from the Riemannian metric and the symplectic form, we discuss the Morse property of the Hamiltonian. If some constraints are put on the circulation vector, then for generic g the n-vortex problem possesses finitely many fixed points and infinitely many periodic orbits. As a byproduct, we exclude the existence of perverse choreography.

Marian Gidea, Yeshiva University
Mathematical models for the Sun-Jupiter-Hektor-Skamandrios system.
Thursday, June 23rd, 12:10-12:50

Professor Gidea presented models of several aspects of the system consisting of the Sun, Jupiter, Jupiter's Trojan asteroid Hektor, and Hektor's moonlet Skamandrios. First, we describe a method to obtain analytical approximations of Hektor's shape. Second, we find triangular central configurations of three oblate bodies.Third, considering the J2 approximation of Hektor's gravitational potential, we derive the Hill four-body problem and compute its equilibria and invariant manifolds. Fourth, we show that the Hill four-body problem is branch regularizable. This is based on joint works with E. Belbruno, J. Burgos-García, A. Celletti, C. Gales, W-T Lam, and F. Zypman.

Connor Jackman, CIMAT
Scaling symmetries and contact reduction.
Thursday, June 23rd, 12:50-13:30

A Hamiltonian system admitting a 'scaling symmetry' may be reduced to a contact Hamiltonian system on a scale reduced space. Prof. Jackman explained this reduction and consider some examples (central force problems and McGehee blow-up). As well, we will see how the scale reduced systems admit variational descriptions as Herglotz-Lagrangian systems. This is joint work with Alessandro Bravetti (IIMAS) and David Sloan (Lancaster).

Guglielmo Feltrin, University of Udine
Bifurcation of closed orbits for central force problems and an application to a restricted 3-body problem.
Friday, June 24th, 9:00-9:40

In this talk, Profesor Feltrin deals with a perturbed central force problem of the form

$$\ddot{x} = V'(|x|)\frac{x}{|x|} + \varepsilon\,\nabla_x U(t,x), \quad x \in \mathbb{R}^2 \setminus \{0\},$$

where $\varepsilon \in \mathbb{R}$ is a small parameter, $V\colon (0,+\infty) \to \mathbb{R}$ and $U\colon \mathbb{R} \times (\mathbb{R}^2 \setminus \{0\}) \to \mathbb{R}$ are smooth functions, and U is τ-periodic in the first variable. Based on the introduction of suitable time-maps (the radial period and the apsidal angle) for the unperturbed problem ($\varepsilon = 0$) and of an associated non-degeneracy condition, we apply a higher-dimensional version of the Poincaré–Birkhoff fixed point theorem to prove the existence of non-circular τ-periodic solutions bifurcating from invariant tori at $\varepsilon = 0$. Our new non-degeneracy condition is nothing but an equivalent formulation of the usual one, having however the advantage of depending only on the potential V and, instead, not requiring the explicit knowledge of the Hamiltonian in action-angle coordinates.

Next, we show that this non-degeneracy condition is satisfied for some concrete examples of physical interest (including the homogeneous potential $V(r) = \kappa/r^\alpha$ for $\alpha \in (-\infty, 2) \setminus \{-2, 0, 1\}$). At last, an application is given to a restricted 3-body problem with a non-Newtonian interaction.

The talk is based on joint work with Alberto Boscaggin and Walter Dambrosio (University of Torino).

Ernesto Pérez-Chavela, ITAM
A new method to study relative equilibria on the sphere $\mathbb{S}^2$. Friday, June 24, 9:40 - 10:20

The simplest solutions of the N –body problem are those where the mutual distances among the masses remain constant for all time, that is the motions behave as a rigid body. For N = 3 on the Euclidean space it is well known that there are exactly five relative equilibria: three collinear (Euler relative equilibria) and two planar forming an equilateral triangle (Lagrange relative equilibria). In this talk Professor Pérez-Chavela showed a new geometrical method to study relative equilibria when the masses are on a sphere in $\mathbb{R}^3$, when the masses are moving under the influence of a general attractive potential. In this case collinear relative equilibria means that the masses are on the same geodesic, otherwise we call them Lagrange relative equilibria.

Luis Benet, Instituto de Ciencias Físicas, UNAM
Automatic differentiation applied to Near-Earth Objects.
Friday, June 24th, 10:20 - 11:00

In this talk Professor Benet presented the use of high-order automatic differentiation techniques to the determination of the orbit of Near-Earth Objects (NEO). Professor Benet addressed in particular the determination of a non-gravitational acceleration arising from the anisotropic thermal re-emission of absorbed radiation –known as the Yarkovsky effect– for Apophis, a NEO which will have a close approach to the Earth on April 13th, 2029. The Yarkovsky effect is the leading source of uncertainty to predict the orbital motion of NEOs. This is joint work with Jorge A Pérez Hernández.

Abimael Bengochea, ITAM
A restricted four-body problem for the eight figure choreography.
Friday, June 24th, 11:30-12:10

Professor Bengochea introduced a planar restricted four-body problem where a massless particle moves under the gravitational influence due to three bodies following the eight figure choreography. We use reversing symmetries to study certain types of symmetric periodic orbits of this system. The symmetric periodic orbits (initial conditions) were computed by means of solving some boundary value problems.

Alessandra Celletti, Universitá di Roma Tor Vergata *From chaos to KAM tori in rotational dynamics, from machine learning to computer-assisted results.*
Friday, June 24th, 12:10- 12:50

In her talk, Professor Celletti studied regular and chaotic motions a simple problem of rotational dynamics, the spin-orbit problem. In the first part, we implement deep learning techniques to classify the motions starting from time series and without any prior knowledge of the underlying dynamics (joint work with C. Gales, V. Rodriguez, M. Vasile).

In the second part, Prof. Celletti generalized the model by including also dissipative effects due to the tidal torque and we construct invariant tori by means of a KAM theorem for conformally symplectic systems. The existence of the tori is validated by a computer-assisted technique (joint work with R. Calleja, J.Gimeno, R. de la Llave).