Snake graphs associated to punctured orbifolds

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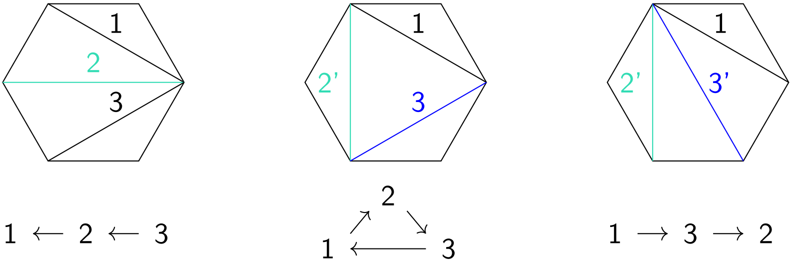
Outline

1. Background on cluster algebras from surfaces and snake graphs
2. Generalized cluster algebras from orbifolds
3. Results for unpunctured orbifolds
4. Progress for punctured orbifolds

Data from a Triangulation

Recall: We get a quiver Q_T from a triangulation T by including

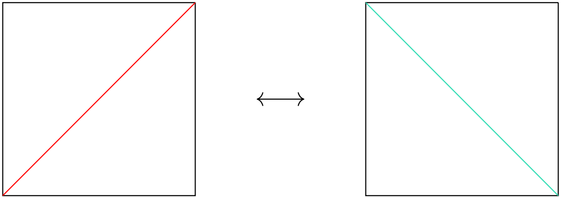
- a vertex i for each $\tau_i \in T$ and
- an arrow $i \to j$ if τ_j immediately follows τ_i in counterclockwise order



Cluster Algebras from Surfaces

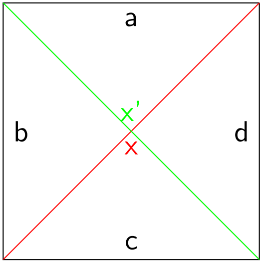
By Fomin-Shapiro-Thurston and Fomin-Thurston, we have bijections

Surface with marked points (S, M)	Cluster Algebra $A(S, M)$
Triangulation of (S, M)	Cluster in $A(S, M)$
(Tagged) Arc on (S, M)	Cluster variable in $A(S, M)$
Flipping arcs	Mutation



Cluster Algebras from Surfaces

We can recover exchange relations for flipping of arcs by iterated use of the Ptolemy Relation:



$$xx' = Y_1ac + Y_2bd$$

y-variables by Laminations

- We keep track of y-variables by placing a *(multi)-lamination* on the surface.
- A lamination is a set of curves whose end points lie on $S \setminus M$.

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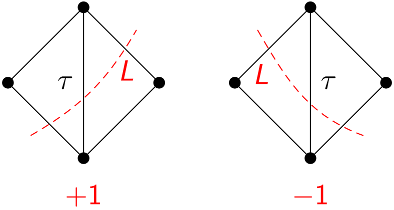
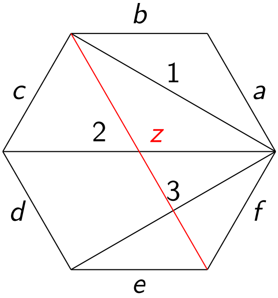
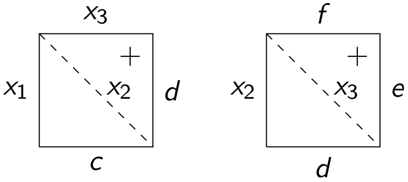
- We keep track of y-variables by placing a *(multi)-lamination* on the surface.
- A lamination is a set of curves whose end points lie on $S \backslash M$.
- The shear coordinate $b_\tau(L, T)$ is calculated by looking at intersections of L and T near τ



Snake Graphs

- Musiker, Schiffler, and Williams developed snake graphs as a way to directly compute the cluster variable from an arc on a surface.
- This construction proves positivity for cluster algebras from surfaces.





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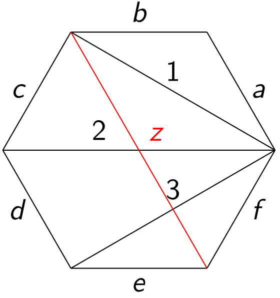

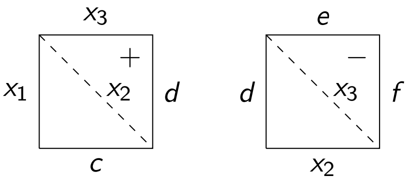

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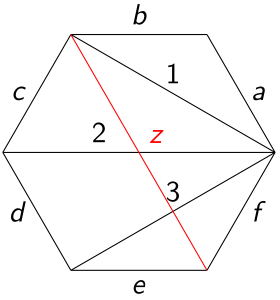

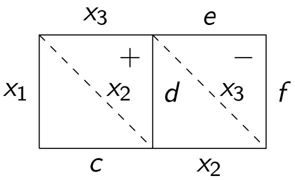

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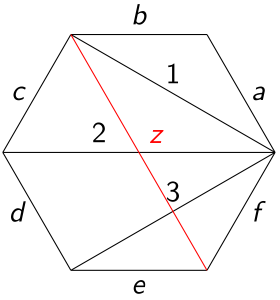

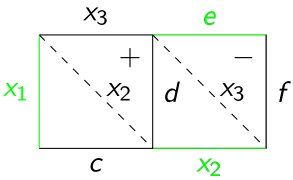

$$z = \frac{1}{x_2 x_3}(e x_1 x_2$$

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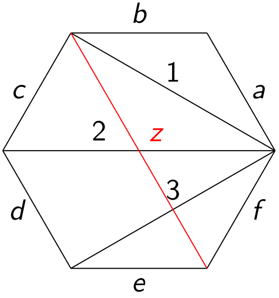

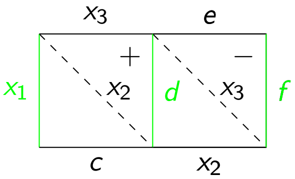

$$z = \frac{1}{x_2 x_3}(e x_1 x_2 + y_3 d f x_1$$

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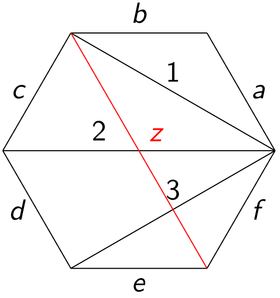

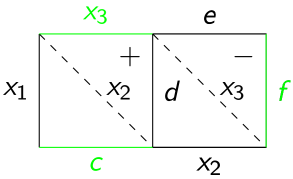

$$z = \frac{1}{x_2 x_3}(ex_1x_2 + y_3 dfx_1 + y_2y_3 cfx_3)$$

Snake Graphs - Theorem statement

Cluster Algebra Expansion Formula [MSW 2009]

Let A be the cluster algebra from the surface (S, M) with triangulation T. Let γ be an arc on S, and let x_γ be the cluster variable from γ. Then,

$$[x_\gamma]^A_{\Sigma_T} = \frac{1}{\mathrm{cross}(T, \gamma)} \sum_P x(P) y(P)$$

where the summation is indexed by perfect matchings of $G_{T,\gamma}$.

- $\mathrm{cross}(T, \gamma)$ is the monomial corresponding to the arcs of T which γ crosses
- We determine $x(P)$ by the weights of edges in the perfect matching P
- We can determine $y(P)$ by establishing a *minimal matching*, M and taking the symmetric difference of P and M.

Generalized Cluster Algebras

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- In order to record these exchange relations, our seeds contains exchange polynomials, $Z_i(u) = z_{i,0} + z_{i,1}u + \cdots + z_{i,d_i}u^{d_i}$ where $z_{i,0} = z_{i,d_i} = 1$ and $d_i \geq 1$.

Generalized Cluster Algebras - Mutation

Mutation μ_k of a generalized seed is given by

$$\mu_k(\{x_1, \dots, x_n\}, \{y_1, \dots, y_n\}, B, \mathbf{d}) = (\{x'_1, \dots, x'_n\}, \{y'_1, \dots, y'_n\}, B', \mathbf{d})$$

where

$$x'_j = \begin{cases} x_j & j \neq k \\ \frac{u_k^{d_k} + z_{k,1} u_k^{d_k - 1} v_k + \cdots + z_{k,d_k-1} u_k v_k^{d_k - 1} + v_k^{d_k}}{x_k} & j = k, \end{cases}$$

where $u_k = y_k \prod x_i^{[b_{i,k}]_+}$ and $v_k = \prod x_i^{[-b_{i,k}]_+}$

$$y'_j = \begin{cases} y_k^{-1} & j = k \\ y_j (y_k^{d_k})^{[b_{k,j}]_+} & j \neq k \end{cases}$$

and $B' = (b'_{i,j})$

$$b'_{i,j} = \begin{cases} -b_{i,j} & i = k \text{ or } j = k \\ b_{i,j} + \operatorname{sgn}(b_{i,k}) d_k [b_{i,k} b_{k,j}]_+ & \text{otherwise} \end{cases}$$

Generalized Cluster Algebras - Example

Let $Z_1(u) = Z_2(u) = u + 1$ and $Z_3(u) = u^2 + \sqrt{2}u + 1$. So $d_1 = d_2 = 1$ and $d_3 = 2$.

Example

$$\{x_1, x_2, x_3\} \overset{\mu_2}{\leftrightarrow} \{x_1, \frac{x_1 + y_2x_3}{x_2}, x_3\} \overset{\mu_3}{\leftrightarrow} \{x_1, \frac{x_1 + y_2x_3}{x_2}, \frac{x_1^2x_2^2 + \sqrt{2}y_3x_1x_2\,(x_1 + y_2x_3) + y_3^2(x_1^2 + 2y_2x_1x_3 + y_2^2x_3^2)}{x_2^2x_3}\}$$

$$\{y_1, y_2, y_3\} \overset{\mu_2}{\leftrightarrow} \{y_1y_2, \frac{1}{y_2}, y_3\} \overset{\mu_3}{\leftrightarrow} \{y_1y_2y_3^2, \frac{1}{y_2}, \frac{1}{y_3}\}$$

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \overset{\mu_2}{\leftrightarrow} \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \overset{\mu_3}{\leftrightarrow} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

Geometry behind generalized cluster algebras - Orbifolds

- Chekhov and Shapiro's original definition of generalized cluster algebras was motivated by orbifolds.
- We will think of an orbifold $\mathcal{O} = (S, M, Q)$ as a surface with a set of special points, Q, called orbifold points.

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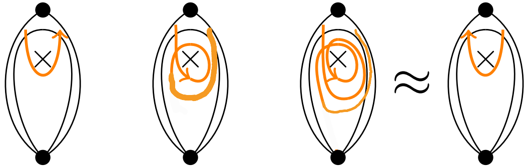
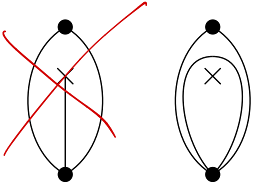
- Chekhov and Shapiro's original definition of generalized cluster algebras was motivated by orbifolds.
- We will think of an orbifold $\mathcal{O} = (S, M, Q)$ as a surface with a set of special points, Q, called orbifold points.
- Each orbifold point will come with an *order*, p, where $p \in \mathbb{Z}$ and $p \geq 2$.
- The order of an orbifold point tells us how many times an arc can wind around it.



Triangulations of Orbifolds

When triangulating an orbifold, we include exactly one *pending arc* to each orbifold point.



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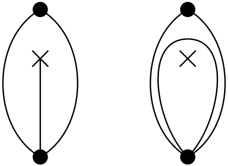
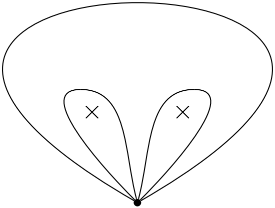
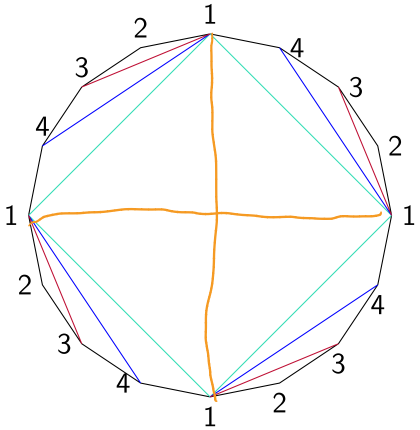


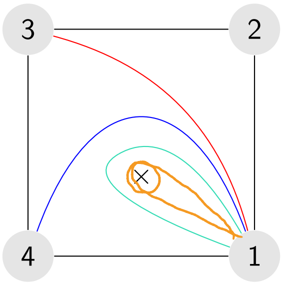
Each pending arc will be enclosed in a bigon. The other sides of the bigon can be standard or pending.



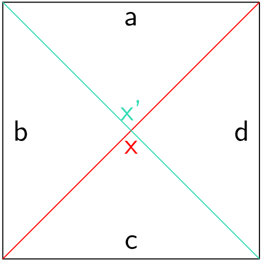
Moving Towards Orbifolds

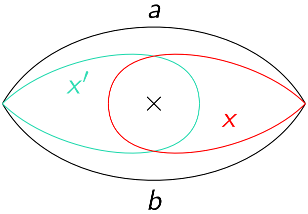
Here is an example of a disk with four marked points on boundary and one orbifold point of order 4 as well as a cover.





Pending arcs have a 3-term Ptolemy-like relation. Let $\lambda_p := 2\cos(\pi/p)$.



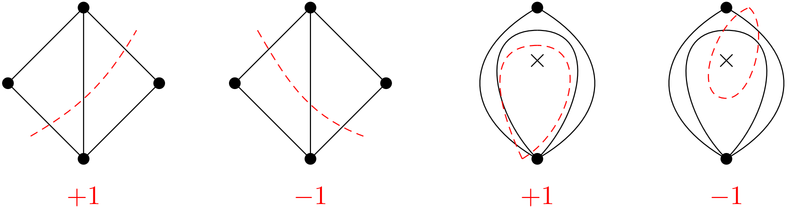


$xx' = Y_1 ac + Y_2 bd$

p	λ_p
3	1
4	$\sqrt{2}$
5	ϕ
6	$\sqrt{3}$

$xx' = Y_1 a^2 + \lambda_p Y_2 ab + Y_3 b^2$

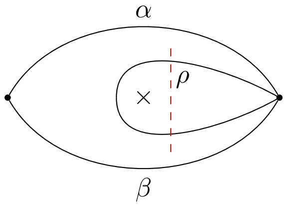
Laminations from Orbifold



Proposition

[B.-Kelley, 2020] The shear-coordinate rules for an orbifold agree with generalized cluster mutation of y-variables.

Snake graphs from orbifolds



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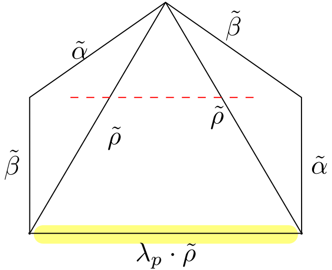
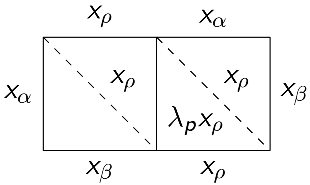
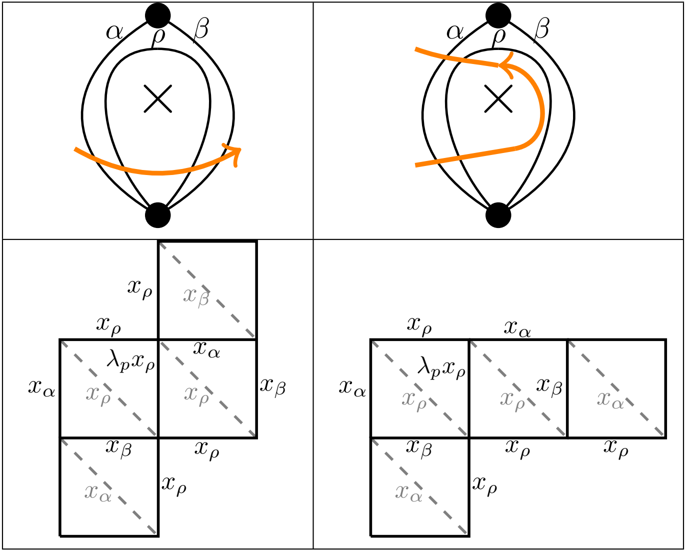




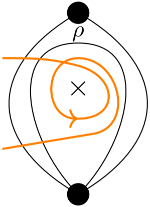
$$\frac{1}{x_\rho^2}\left(x_\alpha^2 x_\rho + y_\rho \lambda_p x_\alpha x_\beta x_\rho + y_\rho^2 x_\beta^2 x_\rho\right)$$

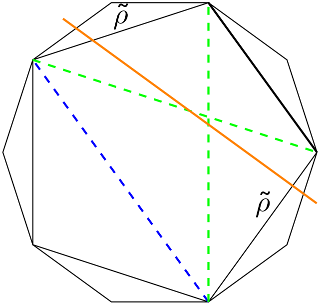
Ordinary Arc puzzle pieces



Looking towards generalized arcs

We must think harder about generalized arcs as below.





Chebyshev Polynomials

Define a normalized family of Chebyshev polynomials of the second kind by

$$U_0(x) = 1 \quad U_1(x) = x \quad U_k(x) = xU_{k-1}(x) - U_{k-2}(x) \text{ for } k > 1$$

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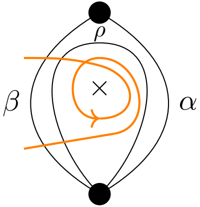
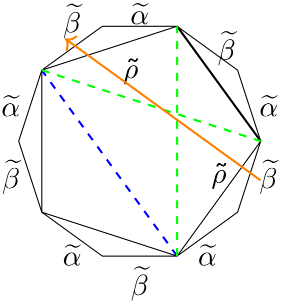
Let a k-diagonal in a polygon be one that skips k vertices.

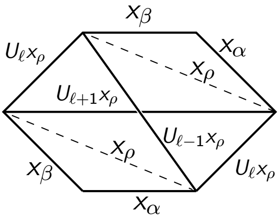
Lemma (Lang)

If P is a regular polygon with p sides, each of length s, a k-diagonal in P has length $U_k(\lambda_p)s$.

An extra edge!

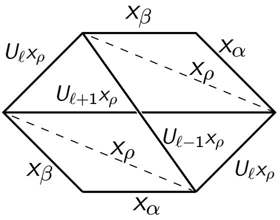






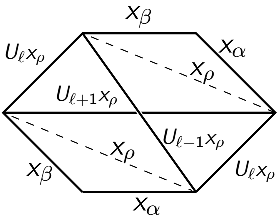
$U_k x_\rho$ stands for $U_k(\lambda_p) x_\rho$

Hexagonal Tile



$$\frac{1}{x_\rho^2}\left(U_\ell(\lambda_\rho)\, x_\alpha^2 x_\rho + y_\rho \left(U_{\ell+1}(\lambda_\rho) + U_{\ell-1}(\lambda_\rho)\right) x_\alpha x_\rho x_\beta + y_\rho^2 U_\ell(\lambda_\rho)\, x_\beta^2 x_\rho\right)$$

Hexagonal Tile



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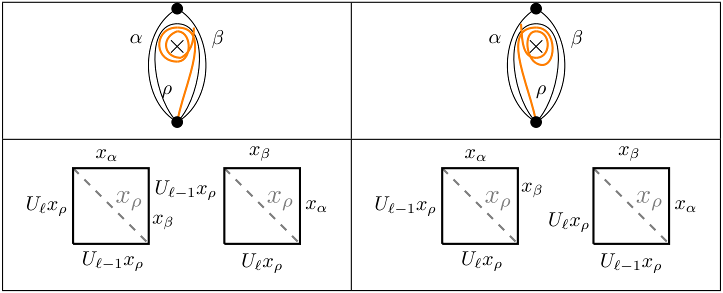
If $\ell = 0$ or $\ell = p - 2$ we recover the expansion of a standard two tile snake graph.

One more case



Leftmost tile:

$$\frac{U_{\ell-1}(\lambda_p)x_\rho x_\alpha + U_\ell(\lambda_p)y_\rho x_\rho x_\beta}{x_\rho} = U_{\ell-1}(\lambda_p)x_\alpha + U_\ell(\lambda_p)y_\rho x_\beta$$

Results for unpunctured case

Theorem (B.-Kelley, 2020)

Let A be the generalized cluster algebra from the unpunctured orbifold $\mathcal{O} = (S, M, Q)$ with triangulation T. Let γ be an arc on $\mathcal{O}$. Then,

$$[x_\gamma]^A_{\Sigma_T} = \frac{1}{cross(\gamma, T)} \sum_P x(P)y(P)$$

where the summation is indexed by perfect matchings of $G_{\gamma,T}$.

Corollary

The coefficients of $[x_\gamma]^A_{\Sigma_T}$ are positive.

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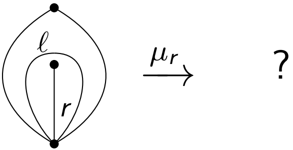
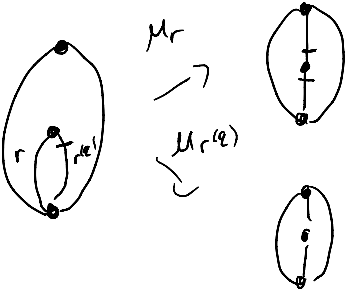
We also show that the snake graph map respects *skein relations*.

Tagged Arcs

When there are *punctures* in the surface, i.e. marked points on the interior, we must consider tagged arcs.





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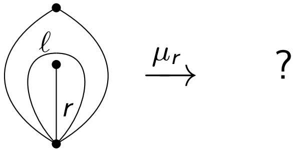

Two tagged arcs γ_1, γ_2 are *compatible* if

- The untagged versions γ_1^0 and γ_2^0 do not cross,
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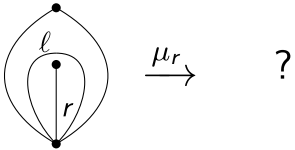

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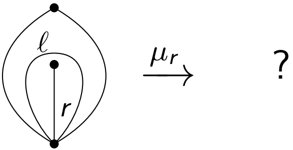
- if $\gamma_1^0 = \gamma_2^0$, at least one endpoint of γ_1 is tagged the same way as γ_2, and
- If $\gamma_1^0 \neq \gamma_2^0$, and these share an endpoint p, then they are tagged in the same way at p.

Tagged Arcs

When there are *punctures* in the surface, i.e. marked points on the interior, we must consider tagged arcs.



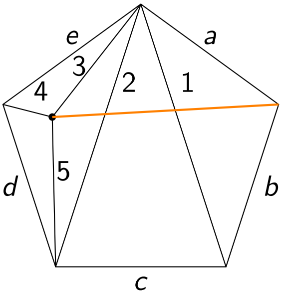
Two tagged arcs γ_1, γ_2 are *compatible* if

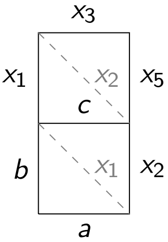
- The untagged versions γ_1^0 and γ_2^0 do not cross,
- if $\gamma_1^0 = \gamma_2^0$, at least one endpoint of γ_1 is tagged the same way as γ_2, and
- If $\gamma_1^0 \neq \gamma_2^0$, and these share an endpoint p, then they are tagged in the same way at p.

Slogan: $x_\ell = x_r x_{r^{(p)}}$

Combinatorial Expansions of Tagged Arcs

- Musiker, Schiffler, and Williams gave expansions for $x_{\gamma^{(p)}}$ and $x_{\gamma^{(pq)}}$ via "γ-symmetric" and "γ-compatible" matchings.
- Wilson introduced a different formulation via *loop graphs*.





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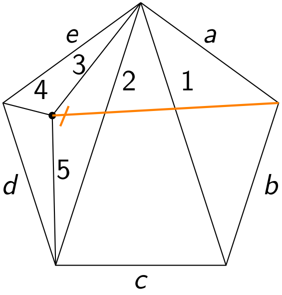

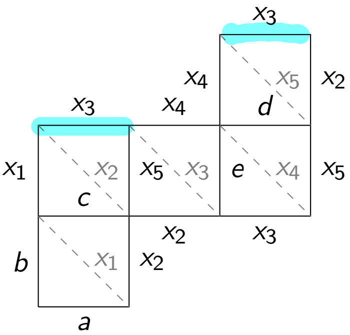

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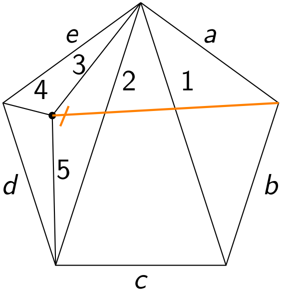
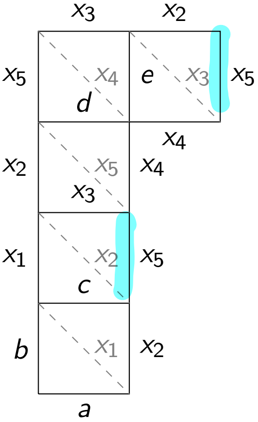
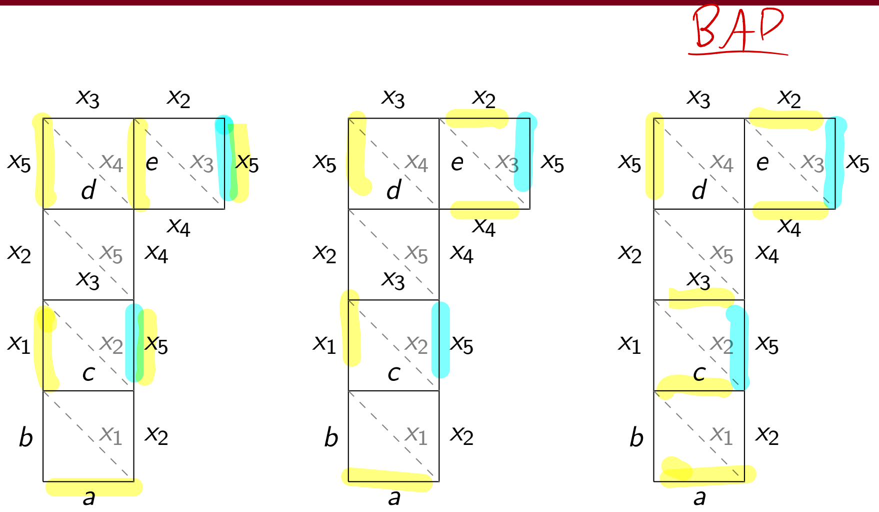


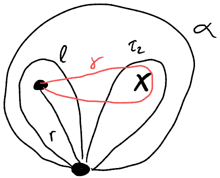


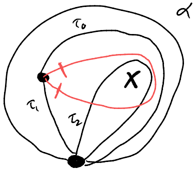
Good matchings of loop graphs

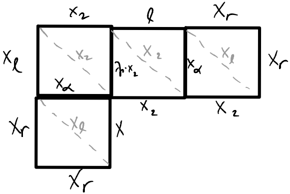


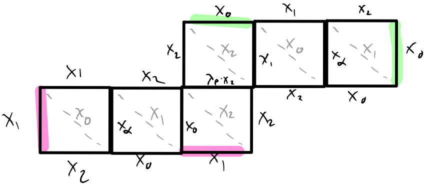
Progress in punctured case

We have all the pieces to extend our expansion formula to ordinary tagged arcs on a punctured orbifold.



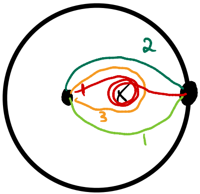


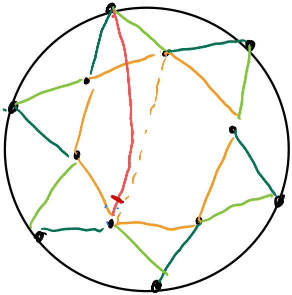


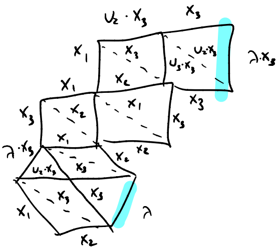


Progress in punctured case

To prove that our choices of expansion for generalized arcs are consistent, we want to show that these satisfy skein relations.







Skein Relations Example

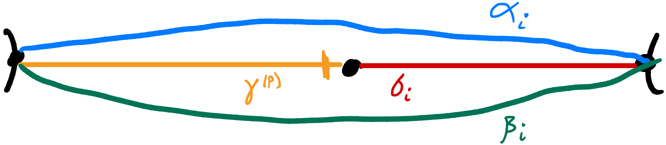
Given an arc γ, let x_γ be the element of $A(S, M)$ associated to γ and let $\chi(\gamma) = \frac{1}{\text{cross}(\gamma, T)} \sum_P x(P)y(P)$.

Proposition (B.-Kang-K., 2022+)

Given orbifold $\mathcal{O}$ with initial trinagulation T, let $\sigma_i, \gamma^{(p)}$ be as below. Let x_γ be the corresponding element of $\mathcal{A}(T)$ and let $\chi(\gamma_i)$ be output of snake graph map. Then,

$$x_{\gamma^{(p)}} x_{\sigma_i} = Y_\alpha x_{\alpha_i} + x_{\beta_i} \quad \text{and} \quad \chi(\gamma^{(p)})\chi(\sigma_i) = Y_\alpha \chi(\alpha_i) + \chi(\beta_i)$$

where Y_α is determined by spokes to p not crossed by α



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Thank you

Thank you for listening!