Bases for cluster algebras, in honour of B. Leclerc, Oaxaca, 2022/09/26, 09:00–09:50

The blue vs. red game and applications

Notes at bit.ly/kellersnotes

Plan: 1. The blue vs. red game [Any resemblance to American politics is purely coincidental!]

2. Relative cluster categories

3. Application: braid subgroup actions

4. Why does it work? The Frobenius case

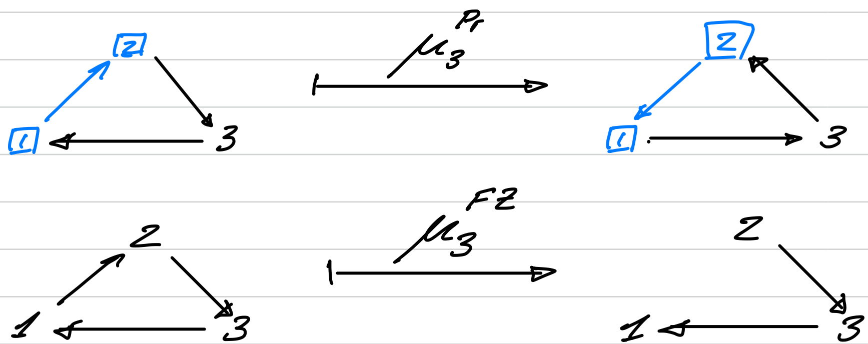
1. The blue vs. red game

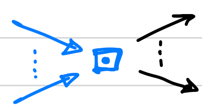
Let (Q, F) be an ice quiver, i.e. a finite quiver Q w/o loops nor 2-cycles with a frozen subquiver $F \subseteq Q$, e.g.

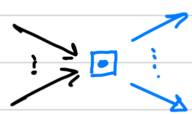
$$\begin{tikzcd} & \boxed{2} \arrow[dr] & \\ \boxed{1} \arrow[ur] & & 3 \arrow[ll] \end{tikzcd}$$

M. Pressland (2018) has extended Fomin-Zelevinsky's mutation rule (2002) so as to take frozen arrows into account.

Example:

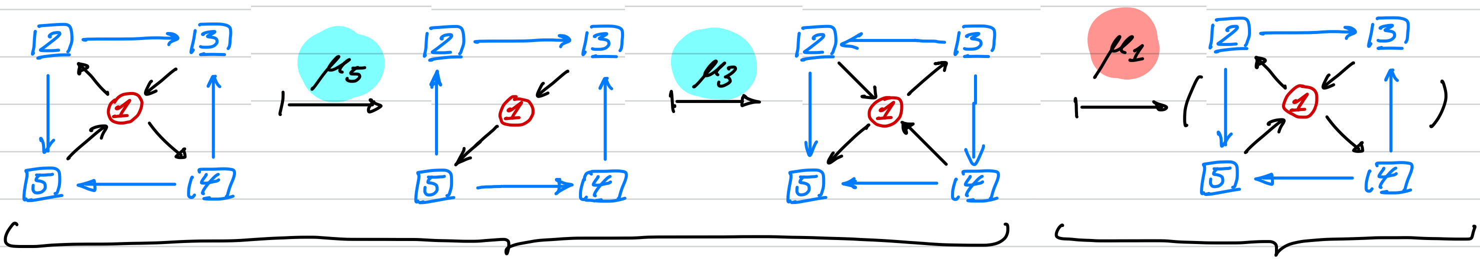


Yilin Wu (2021) observed that Pressland's rule also makes sense for mutations at frozen sinks 

and frozen sources 

cf. Fraser – Sherman-Bennett (06/2020).

Example 1 of the blue vs. red game:

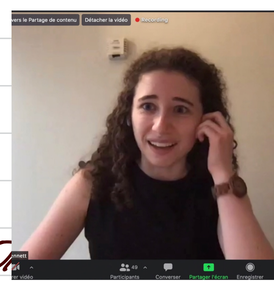


① Reform phase: Perform a sequence $\underline{i}$ of mut. at frozen sinks/sources.

② Counter-reform phase: Perform a seq. of mut. $\underline{k}$ at non frozen vertices to undo ① at least up to an isom. φ.

Matthew Pressland

Chris Fraser



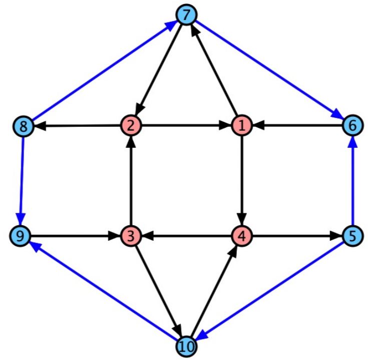
Melissa Sherman-Bennet

Yilin Wu

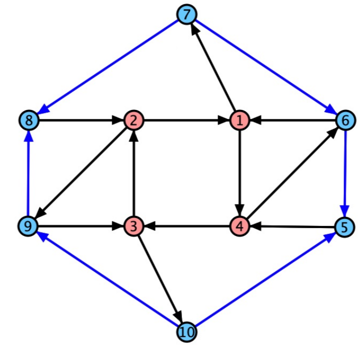
describes $\widetilde{Gr}(3,6)$

Example 2 :

Reform

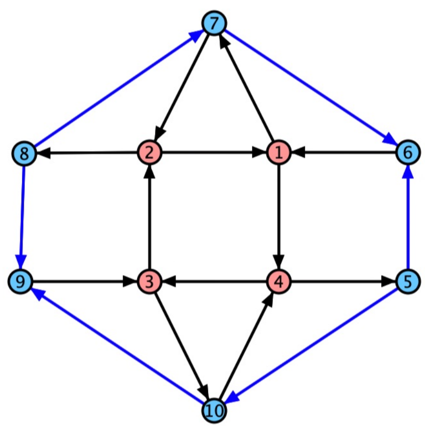


$\xrightarrow[\underline{i}]{5,\ 8}$

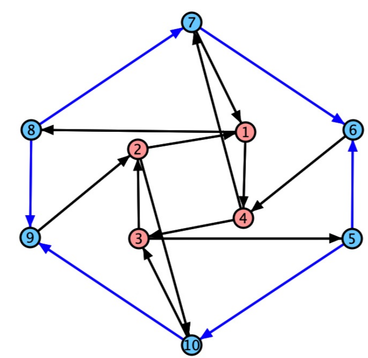


$\underline{k}\ \Big\downarrow\ 1, 3, 2, 4$

Counter-reform



$\xleftarrow[(1\ 2\ 3\ 4)]{\underset{\sim}{\varphi}}$



2. Relative cluster categories

Let (Q,F) be an ice quiver.

The **braid group** associated with F is

$$Br_F = \langle \sigma_j \mid j \in F_0,\ \sigma_j \sigma_\ell \sigma_j = \sigma_\ell \sigma_j \sigma_\ell \text{ if } \exists\, 1\ \ell \to j \text{ or } j \to \ell \rangle$$

The simples S_i, $i \in F_0$, are spherical objects in $\operatorname{per} \Pi$ = perfect derived category,

where Π is the 2-Calabi-Yau completion of kF (Π = preproj. alg. of F if F con. non Dynkin).

Via spherical twists at the S_i, Br_F acts on $\operatorname{per} \Pi$ (Seidel-Thomas 2001),

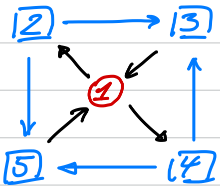
Aim: Make a subgroup $G \subseteq Br_F$ act on the

relative cluster category $\mathcal{C}_{Q,F,W}$ (Yilin Wu '21, '22).

suitable potential

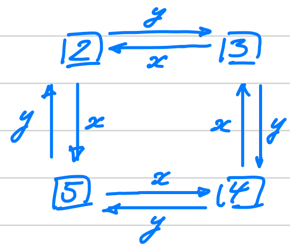
Examples of relative cluster categories:

1)



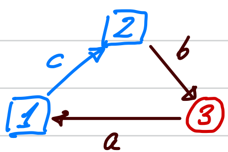
$$W = \sum \triangle - \sum \triangle \implies \mathcal{C}_{Q,F,W} \simeq \mathcal{D}^b(\mathrm{mod}\, B)$$

B = boundary algebra =



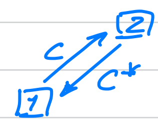
subject to $xy = yx$ and $x^2 = y^2$.

2)



$$W = abc \implies \mathcal{C}_{Q,F,W} \simeq \mathcal{D}^b(\mathrm{mod}\, B)$$

B = boundary algebra = preproj. alg.



, $cc^* = 0 = c^*c$.

General construction of $\mathcal{C}_{Q,F,W}$:

Π = 2-Calabi-Yau completion of kF ($\Rightarrow H^0\Pi$ = preproj. alg. of F)

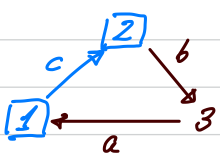
$\big\downarrow G$ = Ginzburg morphism

Γ = relative Ginzburg dg algebra of (Q, F, W)

$$\mathcal{C}_{Q,F,W} = \operatorname{per}\Gamma \,/\, \operatorname{thick}(S_i \mid i \text{ non frozen})$$

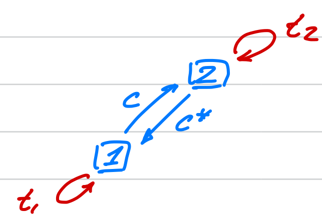
B = boundary algebra $= e H^0\Gamma e$, $e = \sum_{i \text{ frozen}} e_i$, usually $\mathcal{D}^b(\operatorname{mod} B) \neq \mathcal{C}_{Q,F,W}$!

Example:



$W = abc \quad \Longrightarrow$

Π :

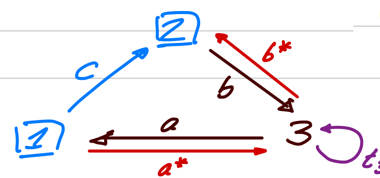


$dt_1 = cc^*$

$dt_2 = -c^*c$

$$\Big\downarrow G$$

Γ :



$da^* = \partial_a W = bc$

$db^* = \partial_b W = ca$

$dt_3 = bb^* - a^*a$

$$\begin{aligned} G: \; t_1 &\longmapsto aa^*, & t_2 &\longmapsto -b^*b \\ c &\longmapsto c, & c^* &\longmapsto \partial_c W \end{aligned}$$

3. Application: braid subgroup actions

reform

counterreform

Keep the notations (Q,F) and $(\underline{i}, \underline{k}, \varphi)$ as in the blue vs. red game.

Define $\beta(\underline{i}) = \sigma_{i_1}^{\varepsilon_1} \sigma_{i_2}^{\varepsilon_2} \cdots \sigma_{i_\ell}^{\varepsilon_\ell} \in Br_F$, where $\underline{i} = (i_1, \ldots, i_\ell)$, $\varepsilon_j = \begin{cases} 1 & i_j \text{ fr. source} \\ -1 & i_j \text{ fr. sink} \end{cases}$

Thm 1: Suppose that (Q,F) admits a potential W which is

a) consistent (i.e. $\Gamma \xrightarrow{\sim} H^0\Gamma$, Γ = rel. Ginzb. alg. of (Q,F,W))

b) epic (i.e. the Ginzburg morphism $\Pi \to \Gamma$ induces a surjection $H^0\Pi \longrightarrow eH^0\Gamma e = \mathcal{B}$ = boundary algebra)

c) preserved under $(\underline{i}, \underline{k}, \varphi)$ (up to right equivalence)

Then the canonical autoequivalence $\Phi(\underline{i},\underline{k},\varphi) \circlearrowright \mathcal{C}_{Q,F,W}$ (Yilin Wu '21) only depends on $\beta(\underline{i}) \in Br_F$.

Cor.: The braid subgroup

$$G = \left\langle \beta(\underline{i}) \;\middle|\; \begin{array}{l} \underline{i} \text{ admits a counterform seq. } \underline{k} \\ \text{and } W \text{ is preserved} \end{array} \right\rangle \subseteq Br_F$$

acts on $\mathcal{C}_{Q,F,W}$ and, via quasi-cluster automorphisms, on the cluster algebra $\mathcal{A}_{Q,F}$ with invertible coefficients.

Confirmed examples: Grassmannian braiding (Fraser '20)

Positroid subbraiding (Fraser-K '22)

Expected examples: Braid subgroup actions due to Bondal '04, Chekhov-Shapiro '20; Fock-Goncharov '06; Goncharov-Shen '16, Inoue-Lam-Pylyavskyy '16, Inoue-Ishibashi-Oya '19, Goncharov-Shen '19; Kashiwara-Kim-Oh-Park '20, ...

4. Why does it work? The Frobenius case

(Q, F, W) an ice quiver with potential, $\mathcal{P} = \mathrm{add}(e\Gamma) \subseteq \mathcal{C}_{Q,F,W}$

$\mathcal{H}$ = **Higgs category** = full subcat. of $\mathcal{C}_{Q,F,W}$ whose objects are the cones $\mathrm{cone}(T_1 \xrightarrow{f} T_0)$ s.th. $T_i \in \mathrm{add}\,\Gamma$ and

$$\begin{tikzcd} T_1 \arrow[r] \arrow[d, "\forall"'] & T_0 \arrow[dl, dashed, "\exists"] \\ P \in \mathcal{P} & \end{tikzcd}$$

Example:

$$\begin{tikzcd} & \boxed{2} \arrow[dr, "b"] & \\ \boxed{1} \arrow[ur, "c"] & & 3 \arrow[ll, "a"] \end{tikzcd}$$

$W = abc \Longrightarrow$

$$\begin{tikzcd} \mathcal{C}_{Q,F,W} \arrow[d, phantom, "\wr"] & \supseteq & \mathcal{H} \arrow[d, phantom, "\wr"] & \supseteq & \mathcal{P} \arrow[d, phantom, "\wr"] \\ \mathcal{D}^b(\mathrm{mod}\,B) & \supseteq & \mathrm{mod}\,B & \supseteq & \mathrm{proj}\,B \end{tikzcd}$$

, B = preproj. alg. of $\boxed{1} \rightarrow \boxed{2}$

Assume that: a) $(\overline{Q}, \overline{W})$ is Jacobi-finite, where $\overline{Q} = Q \setminus F$, $\overline{W} = \dots$

b) (Q, F, W) is consistent, i.e. $\Gamma \xrightarrow[\text{qis}]{} H^0\Gamma$

c) $\mathcal{P}$ is functorially finite in $\mathrm{add}(\Gamma) \subseteq \mathcal{C}_{Q,F,W}$

Examples: Ice quivers with potential associated to Grassmannians, positroids or consistent dimer models on a bordered torus.

Thm: Under these assumptions, $\mathcal{H}$ is a Frobenius exact category with proj.-inj. $\mathcal{P}$ (-Wu) and $\mathcal{H} \hookrightarrow \mathcal{C}_{Q,F,W}$ extends to $\mathcal{D}^b(\mathcal{H}) \xrightarrow{\sim} \mathcal{C}_{Q,F,W}$ (Xiaofa Chen).

Main point of Thm 1: The autoequivalence $\Phi = \Phi(\underline{i}, \underline{k}, \varphi) \circlearrowright \mathcal{C}_{Q,F,W} \simeq \mathcal{D}^b(\mathcal{H})$ is determined by its restriction to the "frozen" subcategory $\mathcal{H}^b(\mathcal{P}) \subseteq \mathcal{D}^b(\mathcal{H})$.

This is natural because each object of $\mathcal{H}$ has a projective resolution.

More precisely:

$$\begin{tikzcd} \mathcal{D}^b(\mathcal{H}) \arrow[loop below] & \mathcal{H}^{-,b}(\mathcal{P}) \arrow[l, "\sim"'] & \supseteq & \mathcal{H}^b(\mathcal{P}) \arrow[loop below] \end{tikzcd}$$

Φ = unique continuous extension of Φ_{res}


Happy birthday, Bernard!