Cichoń's Maximum

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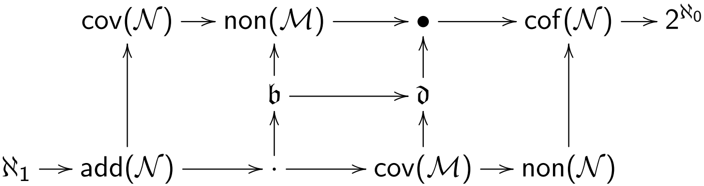
Cichoń's Diagram



- $\mathfrak{b}_{R_1} = \text{add}(\mathcal{N})$, $\mathfrak{d}_{R_1} = \text{cof}(\mathcal{N})$.
- xR_2y iff x is a code for a null set N_x, and $y \notin N_x$. Then $\mathfrak{b}_{R_2} = \text{cov}(\mathcal{N})$ and $\mathfrak{d}_{R_2} = \text{non}(\mathcal{N})$. Well, more or less.
- R_3 is the relation $\leq^*$. Clearly $\mathfrak{b}_{R_3} = \mathfrak{b}$ and $\mathfrak{d}_{R_3} = \mathfrak{d}$.
- xR_4y iff y is a code for a meager set M_y, and $x \in M_y$. Then $\mathfrak{b}_{R_4} = \text{non}(\mathcal{N})$ and $\mathfrak{d}_{R_4} = \text{cov}(\mathcal{N})$.

All entries on the left side will be "unbounding numbers" $\mathfrak{b}_R$ for some relation R, and on the right side "dominating numbers" for the same relation $\mathfrak{d}_R$.

$$\begin{array}{ccccccccccc}
 & & \mathfrak{b}_{R_2} & \longrightarrow & \mathfrak{b}_{R_4} & \longrightarrow & \bullet & \longrightarrow & \mathfrak{d}_{R_1} & \longrightarrow & \mathfrak{d}_{R_0} \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & & & \mathfrak{b}_{R_3} & \longrightarrow & \mathfrak{d}_{R_3} & & & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\aleph_1 & \longrightarrow & \mathfrak{b}_{R_1} & \longrightarrow & \cdot & \longrightarrow & \mathfrak{d}_{R_4} & \longrightarrow & \mathfrak{d}_{R_2} & &
\end{array}$$

(Exercise: Define R_0 and R_1.)

Additivity and Cofinality

For any set S and relation $R \subseteq S \times S$ we define

- $\mathfrak{b}_R$ = the minimal size of an "unbounded" set:

$$\mathfrak{b}_R := \min\{|X| : X \subseteq S, \forall s \in S \; \exists x \in X : \neg(xRs\}$$

- $\mathfrak{d}_R$ = the minimal size of a "dominating" set:

$$\mathfrak{d}_R := \min\{|Y| : Y \subseteq S, \forall s \in S \; \exists y \in Y : \neg(sRY\}$$

(In all cases we consider, this numbers will be well-defined, and usually between $\aleph_1$ and $2^{\aleph_0}$.)
If R is a partial order, and moreover directed without a last element, then $\mathfrak{b}_R$ may be called the unbounding number, the directedness, the completeness or the additivity of R; the cardinal $\mathfrak{d}_R$ is the dominating number or cofinality of R.
If R is moreover linear, then $\mathfrak{b}_R = \mathfrak{d}_R$. (trivial but useful)

An example: Small $\mathfrak{b}$, large $\mathfrak{d}$

After adding Cohen reals $(c_\alpha : \alpha < \tau)$ (τ regular) we have

$$\forall g \ \exists \alpha < \tau : c_\alpha \not\leq^* g$$

hence the Cohen reals are unbounded. They "witness" $\mathfrak{b} \leq \tau$.
But in fact

$$\forall g \ \forall^\infty \alpha < \tau : c_\alpha \not\leq^* g$$

(i.e., the set $\{\alpha < \tau : c_\alpha \leq^* g\}$ is bounded in τ.)
The Cohen reals are a "strong witness" for $\mathfrak{b} \leq \tau$ **and** $\mathfrak{d} \geq \tau$.

Definition (The LCU spectrum (linear cofinal unbounded))

$\tau \in \mathrm{LCU}_R(P)$ iff $\exists (c_\alpha : \alpha < \tau) \ \forall g : \forall^\infty i \ \neg(c_\alpha R g)$.

Note: This implies $\mathfrak{b}_R \leq \tau$ and $\mathfrak{d}_R \geq \tau$.

An example: Large $\mathfrak{b}$, small $\mathfrak{d}$

Let $S \subseteq \delta$, $(w_\alpha : \alpha \in S)$ be cofinal in $[\delta]^{<\lambda}$, with $w_\alpha \subseteq \alpha$.
Consider an iteration $(P_\alpha, Q_\alpha : \alpha < \delta)$, where for all $\alpha \in S$

Q_α adds a real g_α dominating the universe $V^{P_{w_\alpha}}$

The set S is naturally ordered by $\alpha \sqsubseteq \beta \Leftrightarrow w_\alpha \subseteq w_\beta$. Then $(S, \sqsubseteq)$ is $<\lambda$-directed; let $\mu := cof(S, \sqsubseteq)$.
Then P_δ forces $\mathfrak{b} \geq \lambda$ and $\mathfrak{d} \leq \mu$.

Definition (Cone of bounds)

$\mathrm{COB}_R(P, S)$ or $\mathrm{COB}(P; \lambda, \mu)$ or $\mathrm{COB}_R(P, S; \lambda, \mu)$ means: S is a partial order with additivity λ, cofinality μ, and there are names $(c_s : s \in S)$ such that for all names x there is $s_0 \in S$:
$P \Vdash \forall s \geq s_0 : xRc_s$.

Note: This implies $P \Vdash \mathfrak{b}_R \geq \lambda$, $\mathfrak{d}_R \leq \mu$.

Four relations to describe cardinals in Cichoń's Diagram

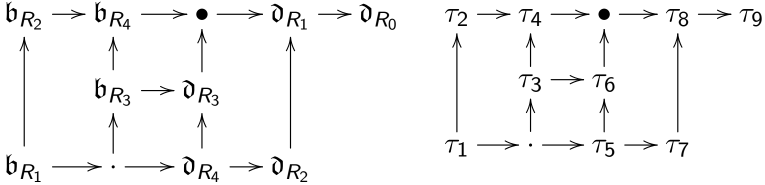
- $\mathfrak{b}_{R_1} = \text{add}(\mathcal{N})$, $\mathfrak{d}_{R_1} = \text{cof}(\mathcal{N})$.
- xR_2y iff x is a code for a null set N_x, and $y \notin N_x$.
  Then $\mathfrak{b}_{R_2} = \text{cov}(\mathcal{N})$ and $\mathfrak{d}_{R_2} = \text{non}(\mathcal{N})$. Well, more or less.
- R_3 is the relation $\leq^*$. Clearly $\mathfrak{b}_{R_3} = \mathfrak{b}$ and $\mathfrak{d}_{R_3} = \mathfrak{d}$.
- xR_4y iff y is a code for a meager set M_y, and $x \in M_y$. Then $\mathfrak{b}_{R_4} = \text{non}(\mathcal{N})$ and $\mathfrak{d}_{R_4} = \text{cov}(\mathcal{N})$.

Will find an iteration P_9 such that $\forall i \in \{0, 1, 2, 3, 4\}$:

- $\tau_i, \tau_{9-i} \in \text{LCU}_i(P_9)$ (so $\mathfrak{b}_{R_i} \leq \tau_i$, $\mathfrak{d}_{R_i} \geq \tau_{9-i}$)
- $\text{COB}_i(P_9, S_i; \tau_i, \tau_{9-i})$ for some S_i. (so $\mathfrak{b}_R \geq \tau_i$, $\mathfrak{d}_R \leq \tau_{9-i}$)

What we want

Writing $\mathfrak{b}_i$ for $\mathfrak{b}_{R_i}$, $\mathfrak{d}_i$ for $\mathfrak{d}_{R_i}$,

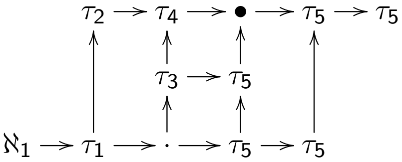


We want to construct a forcing notion P such that:

- $\tau_i, \tau_{9-i} \in \mathrm{LCU}_i(P_\delta)$ (so $\mathfrak{b}_i \le \tau_i$, $\mathfrak{d}_i \ge \tau_{9-i}$)
- $\mathrm{COB}_i(P, S_i; \tau_i, \tau_{9-i})$ for some S_i. (so $\mathfrak{b}_i \ge \tau_i$, $\mathfrak{d}_i \le \tau_{9-i}$)

What we have

We have: Left side with strong witnesses.
Assume GCH and $\aleph_1 < \tau_1 < \tau_2 < \tau_3 < \tau_4 < \tau_5$. Then there is a forcing notion P_5 such that P_5 forces not only this diagram:



but moreover has strong witnesses:

- LCU $_{R_i}(P_5) \supseteq [\tau_i, \tau_5] \cap \mathrm{regular}$.
- COB $_{R_i}(P_5, S; \tau_i, \tau_5)$

Boolean ultrapowers

- Let B be an atomless cBA (complete Boolean algebra), κ^+-cc, $< \kappa$-distributive.
- A BUP-name is a pair (A, x), where A is a maximal antichain, $x : A \to V$ a function. (Write x instead of (A, x).)
  In other words: x is a nice forcing name for an element of V.
- Define Boolean values $[\![x = y]\!]$, $[\![x \in y]\!]$, $[\![\varphi(x)]\!]$ naturally. b
- For any $<\kappa$-complete ultrafilter U, define
  $x \sim_U y \Leftrightarrow [\![x = y]\!] \in U$.
- Let M_- the set of equivalence classes, $\in_-$ the natural relation.
- $(M_-, \in_-)$ is well-founded; collapse it to $(M, \in)$.
- Łoś: $(M, \in) \models \varphi$ iff $[\![\varphi]\!] \in U$.

Boolean ultrapowers, continued

- A BUP-name is a pair (A, x), where A is a maximal antichain, $x : A \to V$ a function. (Write x instead of (A, x).)
  In other words: x is a forcing name for an element of V.
- Let U be a $<\kappa$-complete ultrafilter U; define
  $x \sim_U y \Leftrightarrow [\![x = y]\!] \in U$.
- $(M_-, \in_-)$ (equiv. classes) is WF; collapse it to $(M, \in)$.
- Łoś: $(M, \in) \models \varphi$ iff $[\![\varphi]\!] \in U$.
- $j : V \to M$, using standard names: $j(x) := \check{x}/\sim$.
- Let $A = \{a_\alpha : \alpha < \kappa\}$ be a maximal antichain, disjoint from U.
  Letting $x(a_\alpha) := \alpha$: $\forall \beta < \kappa : j(\beta) = \beta < x < j(\kappa)$. $cp(j) = \kappa$.
- Every element of M can be seen as the "U-average" of κ many elements of V.
  Hence: If $(S, \leq)$ is $\leq\kappa$-directed, then $j``S$ is cofinal in $j(S)$.

Stretching with ultrapowers

$j : V \to M$ with critical point κ. Using an appropriate Boolean algebra, we can make $j(\kappa)$ have arbitrary large cofinality.
(Hint: partial functions from τ to κ, of size $< \kappa$.)
Moreover, M will be σ-closed; so statements about reals (and even about names for reals, by ccc) will be absolute between M and V.
If $M \models j(P) \Vdash \varphi(x)$, then also $V \models j(P) \Vdash \varphi(x)$.

- If $\lambda \in \mathrm{LCU}(P)$ has cofinality $\neq \kappa$, then also $\lambda \in \mathrm{LCU}(j(P))$.
- If $\kappa \in \mathrm{LCU}(P)$, then $j(\kappa) \in \mathrm{LCU}(j(P))$.
- If $\mathrm{COB}(P, S; \lambda, \mu)$, and $\lambda < \kappa$, then $\mathrm{COB}(j(P), j(S); \lambda, j(\mu))$.
- If $\mathrm{COB}(P, S; \lambda, \mu)$, and $\lambda > \kappa$, then $\mathrm{COB}(j(P), j``S; \lambda, \mu)$.

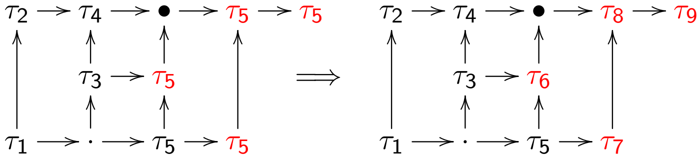
Embeddings $P_5 \xrightarrow{j_6} P_6 \xrightarrow{j_7} P_7 \xrightarrow{j_8} P_8 \xrightarrow{j_9} P_9$

Let j_6 be a BUP, $\tau_3 < \kappa_6 = cp(j_6) < \tau_4$, $cf(j(\kappa)) = \tau_6$.

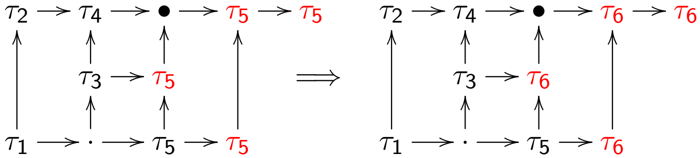
- From $\tau_4, \tau_5 \in \mathrm{LCU}_4(P)$ we conclude $\tau_4, \tau_5 \in \mathrm{LCU}(j(P))$. So $j(P) \Vdash \mathfrak{b}_4 \leq \tau_4, \mathfrak{d}_4 \geq \tau_5$.
- From $\mathrm{COB}_4(P; \tau_4, \tau_5)$ we conclude $\mathrm{COB}_4(j(P); \tau_4, \tau_5)$. (We use $\kappa_6 < \tau_4$ here.) So $j(P) \Vdash \mathfrak{b}_4 \geq \tau_4, \mathfrak{d}_4 \leq \tau_5$.
- From $\tau_3, \kappa, \tau_4 \in \mathrm{LCU}_3(P_5)$ we conclude $\tau_3, \tau_4, \tau_6 \in \mathrm{LCU}_3(j(P))$. So $j(P) \Vdash \mathfrak{b}_3 \leq \tau_4, \mathfrak{d}_3 \geq \tau_6$.
- From $\mathrm{COB}_3(P; \tau_3, \tau_5)$ we conclude $\mathrm{COB}_3(j(P); \tau_3, \tau_6)$. Side computation: $|j(\tau_5)| = |j(\kappa)| = \tau_6$. So $j(P) \Vdash \mathfrak{b}_3 \geq \tau_3, \mathfrak{d}_3 \leq \tau_6$. (We use $\kappa_6 > \tau_3$ here.)
- Similarly for R_2, R_1.

Next step: Apply j_7 to $j_6(P_5)$, with critical point κ_7 between τ_2 and τ_3. Then j_8 and finally j_9 with critical point $\kappa_9 < \tau_1$.

Recall: we want to go from the first diagram to the second one:



A first step:



The LCU table: $P_5 \xrightarrow{j_6} P_6 \xrightarrow{j_7} P_7 \xrightarrow{j_8} P_8 \xrightarrow{j_9} P_9$

$\kappa_9 < \tau_1 < \kappa_8 < \tau_2 < \kappa_7 < \tau_3 < \kappa_6 < \tau_4 < \tau_5 < \cdots < \tau_9$.
$j(\kappa_i) = \tau_i$ for $i = 6, 7, 8, 9$.
Recall: $\lambda \in \mathrm{LCU}_R(P)$ iff $\lambda \in \mathrm{LCU}_R(j(P))$, as long as $cf(\lambda) \neq \kappa$.
But $\kappa \in \mathrm{LCU}(P)$ implies $j(\kappa) \in \mathrm{LCU}(j(P))$.

	R_1	R_2	R_3	R_4
P_5	$[\tau_1, \tau_5]$	$[\tau_2, \tau_5]$	$[\tau_3, \tau_5]$	$[\tau_4, \tau_5]$
P_5	τ_1, κ_8, τ_5	τ_2, κ_7, τ_5	τ_3, κ_6, τ_5	τ_4, τ_5
P_6	τ_1, κ_8, τ_5	τ_2, κ_7, τ_5	τ_3, τ_6	τ_4, τ_5
P_7	τ_1, κ_8, τ_5	τ_2, τ_7	$\tau_3, \tau_6,$	τ_4, τ_5
P_8	τ_1, τ_8	τ_2, τ_7	τ_3, τ_6	τ_4, τ_5

The COB table: $P_5 \xrightarrow{j_6} P_6 \xrightarrow{j_7} P_7 \xrightarrow{j_8} P_8 \xrightarrow{j_9} P_9$

$\kappa_9 < \tau_1 < \kappa_8 < \tau_2 < \kappa_7 < \tau_3 < \kappa_6 < \tau_4 < \tau_5 < \cdots < \tau_9$.
$j(\kappa_i) = \tau_i$ for $i = 6, 7, 8, 9$.
Recall: $\mathrm{COB}_R(P; \lambda, \mu)$ implies $\mathrm{COB}_R(j(P); \lambda, \mu)$ whenever $\kappa < \lambda$.
But for $\kappa > \lambda$ we get only $\mathrm{COB}_R(j(P); \lambda, |j(\mu)|)$.
In the following table, an entry λ/μ in two P_l and column R_i means that P_l forces $\mathrm{COB}_{R_i}(P; \lambda, \mu)$ (i.e., there is an S witnessing COB with additivity λ and cofinality μ.

	R_1	R_2	R_3	R_4
P_5	τ_1/τ_5	τ_2/τ_5	τ_3/τ_5	τ_4/τ_5
P_6	τ_1/τ_5	$\tau_2/,\tau_5$	τ_3/τ_6	τ_4/τ_5
P_7	τ_1/τ_5	τ_2/τ_7	$\tau_3/\tau_6,$	τ_4/τ_5
P_8	τ_1/τ_8	τ_2/τ_7	τ_3/τ_6	τ_4/τ_5

Recall LCU and COB

The plan

We start with a forcing notion P giving "inflated" values to Cichoń's diagram; all different on the left side, and a single (larger) value on the right side.
Then we cleverly construct a model N. The forcing notion $P \cap N$ will force "quite arbitrary" values (smaller than the inflated values) on both sides.

- We use $\{\tau_{\mathrm{left}}, \tau_{\mathrm{right}}\} \in \mathsf{LCU}_R(P \cap N)$ to show $\mathfrak{b}_R \leq \tau_{\mathrm{left}}$ and $\mathfrak{d}_R \geq \tau_{\mathrm{right}}$.
- We use $\mathsf{COB}(P, S; \tau_{\mathrm{left}}, \tau_{\mathrm{right}})$ to show $\mathfrak{b}_R \geq \tau_{\mathrm{left}}$ and $\mathfrak{d}_R \leq \tau_{\mathrm{right}}$.

To achieve this aim, we have to compute/estimate how the values of LCU and COB change when we replace P by $P \cap N$.

Additivity and Cofinality

add$(S) = \mathfrak{b}_S$ = additivity of a (directed) partial order = smallest size of an unbounded (linearly ordered)
$(S) = \mathfrak{d}_S$ = cofinality = smallest size of a dominating set.

Definition

A θ-model is an elementary submodel of "the universe" of cardinality θ, containing $\theta \cup \{\theta\}$ as a subset, and $<\theta$-closed. (Also containining "everything mentioned so far".)
A (λ, θ)-model N is the union of an increasing sequence $(N_\alpha : \alpha < \lambda)$ of θ-models, each containing the sequence of all previous ones.

We will typically have $\lambda \ll \theta$; thus, a (λ, θ)-model is usually only $<\lambda$-closed.