Hilbert's Nullstellensatz and NP-Complete Problems

Susan Margulies

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citing work by N. Alon, S.Buss, J.A. de Loera, R. Impagliazzo, J. Lee, P. Malkin, S. Onn, D.V. Pasechnik, T. Pitassi, P. Pudlák, J. Sgall and **many** others!

Casa Matemática Oaxaca

Theory and Practice of Satisfiability Solving
August 30, 2018

Big Picture Overview

Combinatorial problem (i.e. Partition, graph-k-colorability, matching...)

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Grobner Bases

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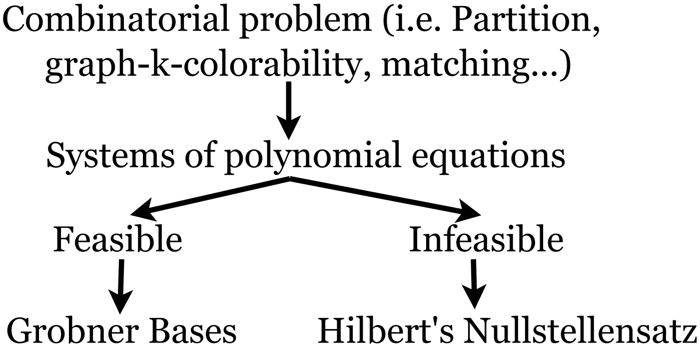

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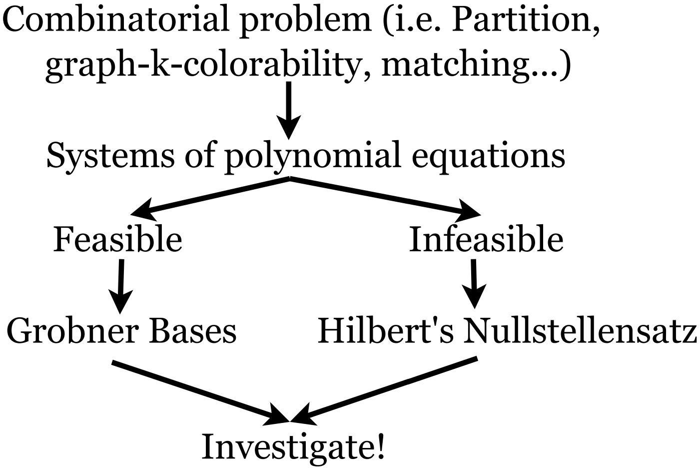

Hilbert's Nullstellensatz

- **Theorem (1893):** Let $\mathbb{K}$ be an algebraically closed field and $f_1, \ldots, f_s$ be polynomials in $\mathbb{K}[x_1, \ldots, x_n]$. Given a system of equations such that $\mathbf{f_1} = \mathbf{f_2} = \cdots = \mathbf{f_s} = \mathbf{0}$, then this system has no solution if and only if there exist polynomials $\beta_{\mathbf{1}}, \ldots, \beta_{\mathbf{s}} \in \mathbb{K}[x_1, \ldots, x_n]$ such that

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- **Definition:** Let $d = \max \left\{ \deg(\beta_1), \deg(\beta_2), \ldots, \deg(\beta_s) \right\}$. Then d is the *degree of the Nullstellensatz certificate*.

-

Nullstellensatz Degree *Upper* Bounds

Recall n is the number of variables, and the number of monomials of degree d in n variables is $\binom{n+d-1}{n-1}$.

- **Theorem:** (Kollár, 1988) The $\deg(\beta_i)$ is bounded by

$$\deg(\beta_i) \leq \Big(\max\big\{3, \max\{\deg(f_i)\}\big\}\Big)^n .$$

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Question: What about lower bounds? How do we find them?

A Too Brief Mention of Highly Significant Papers

- S. Buss, T. Pitassi, *Good Degree Bounds on Nullstellensatz Refutations of the Induction Principle*, Journal of Computer and System Sciences, 1998, 57:162-171.

- R. Impagliazzo, P. Pudlák, J. Sgall, *Lower Bounds for the Polynomial Calculus and Gröbner Basis Algorithm*, J. Computational Complexity, (1999) 8: 127.

Why Combinatorial Problems as Systems of Polynomial Equations?

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- Noga Alon, *Combinatorial Nullstellensatz*, Combinatorics, Probability and Computing 8, 729 (1999)
- Noga Alon (2000): *"Is it possible to modify the algebraic proofs given here so that they yield efficient ways of solving the corresponding algorithmic problems? It seems likely that such algorithms do exist."*

How do we find Nullstellensatz certificates?

- A system of polynomial equations

$$x_1^2 - 1 = 0, \qquad x_1 + x_3 = 0, \qquad x_1 + x_2 = 0, \qquad x_2 + x_3 = 0$$

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1. Construct a hypothetical Nullstellensatz certificate of degree 1

$$1 = \underbrace{(c_0 x_1 + c_1 x_2 + c_2 x_3 + c_3)}_{\beta_1}(x_1^2 - 1) + \underbrace{(c_4 x_1 + c_5 x_2 + c_6 x_3 + c_7)}_{\beta_2}(x_1 + x_2)$$
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2. Expand the hypothetical Nullstellensatz certificate

$$c_0 x_1^3 + c_1 x_1^2 x_2 + c_2 x_1^2 x_3 + (c_3 + c_4 + c_8) x_1^2 + (c_5 + c_{13}) x_2^2 + (c_{10} + c_{14}) x_3^2 +$$
$$(c_4 + c_5 + c_9 + c_{12}) x_1 x_2 + (c_6 + c_8 + c_{10} + c_{12}) x_1 x_3 + (c_6 + c_9 + c_{13} + c_{14}) x_2 x_3 +$$
$$(c_7 + c_{11} - c_0) x_1 + (c_7 + c_{15} - c_1) x_2 + (c_{11} + c_{15} - c_2) x_3 - c_3$$

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3. Extract a *linear* system of equations from expanded certificate

$$c_0 = 0, \qquad \ldots, \qquad c_3 + c_4 + c_8 = 0, \qquad c_{11} + c_{15} - c_2 = 0, \qquad -c_3 = 1$$

NulLA running on a particular instance:

	c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}	c_{15}	
x_1^3	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$x_1^2x_2$	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$x_1^2x_3$	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x_1^2	0	0	0	1	1	0	0	0	1	0	0	0	0	0	0	0	0
x_2^2	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0
x_3^2	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0
x_1x_2	0	0	0	0	1	1	0	0	0	1	0	0	1	0	0	0	0
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x_3	0	0	-1	0	0	0	0	0	0	0	0	1	0	0	0	1	0
1	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	1

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4. Solve the linear system, and assemble the certificate

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5. Otherwise, increment the degree and repeat.

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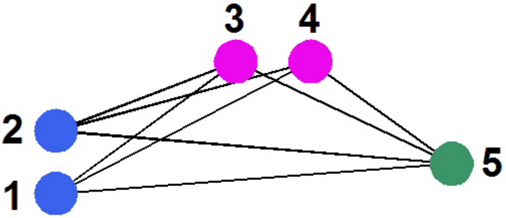
- **Turán Graph** $T(5,3)$**:**



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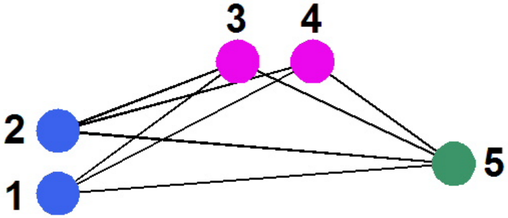
- **Turán Graph $T(5,3)$:** $\alpha\big(T(5,3)\big) = 2$.



Independent Set as a System of Polynomial Equations (L. Lovász)

Given a graph G and an integer k:

- one **variable** per **vertex**: $x_1, \ldots, x_n$
- For every vertex $i = 1, \ldots, n$, let $x_i^2 - x_i = 0$.
- For every edge $(i,j) \in E(G)$, let $x_i x_j = 0$.
- Finally, let

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- **Theorem:** Let G be a graph, k an integer, encoded as the above $(n + m + 1)$ system of equations. Then this system has a solution if and only if G has an independent set of size k.

Turán Graph $T(5,3)$: $\Longrightarrow$ System of Polynomial Equations

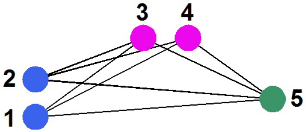
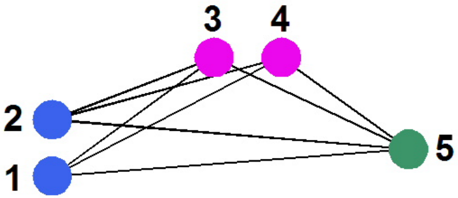


Figure: Does $T(5,3)$ have an independent set of size 3?

$$x_1x_3 = 0,\ x_1x_4 = 0,\ x_1x_5 = 0,\ x_2x_3 = 0, \quad x_1^2 - x_1 = 0,\ x_2^2 - x_2 = 0$$
$$x_2x_4 = 0,\ x_2x_5 = 0,\ x_3x_5 = 0,\ x_4x_5 = 0, \quad x_3^2 - x_3 = 0,\ x_4^2 - x_4 = 0$$
$$x_1 + x_3 + x_5 + x_2 + x_4 - 3 = 0, \quad x_5^2 - x_5 = 0$$

- **Remark:** Since $T(5,3)$ has **no** independent set of size 3, this system of polynomial equations is *infeasible*.

Turán Graph $T(5,3)$: Reduced Certificate Example



$$1 = \left(\frac{x_1x_2 + x_3x_4}{12} - \frac{x_1+x_3+x_5+x_2+x_4}{12} - \frac{1}{4}\right)(x_1+x_3+x_5+x_2+x_4-4) +$$
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Nullstellensatz certificates of Independent Set have Large Degree and are Dense

- **Theorem (J. De Loera, J. Lee, S.M., S. Onn, 2007):** For a graph G, a minimum-degree Nullstellensatz certificate for the non-existence of a independent set of size greater than $\alpha(G)$ has degree equal to $\alpha(G)$ and contains at least one term for every independent set in G.

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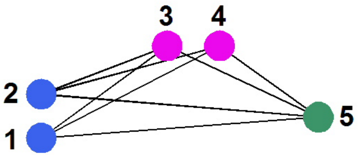

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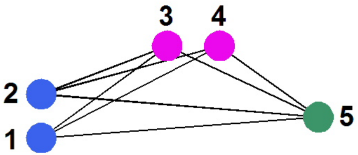

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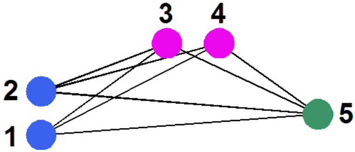

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Question:

Do the actual *numbers* within the Nullstellensatz certificates likewise have a combinatorial interpretation?

Partition Problem: Definition and Example

- **Partition:** Given set of integers $W = \{w_1, \ldots, w_n\}$, can W be partitioned into two sets, S and $W \setminus S$ such that

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- The **Partition** problem is NP-complete.

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Question: Let $W = \{1, 3, 5, 2\}$. Is W partitionable?

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$$\begin{aligned}
1 = &\left(-\frac{155}{693} + \frac{842}{3465}x_2x_3 - \frac{188}{693}x_2x_4 + \frac{908}{3465}x_3x_4\right)(\mathbf{x_1^2 - 1}) \\
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&+ \left(-\frac{467}{693} + \frac{842}{693}x_1x_2 + \frac{908}{693}x_1x_4 + \frac{292}{693}x_2x_4\right)(\mathbf{x_3^2 - 1}) \\
&+ \left(-\frac{68}{693} - \frac{376}{693}x_1x_2 + \frac{1816}{3465}x_1x_3 + \frac{584}{3465}x_2x_3\right)(\mathbf{x_4^2 - 1}) \\
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Let S_k^n denote the set of k-subsets of $\{1, \ldots, n\}$ $\left(\text{ i.e., } |S_k^n| = \binom{n}{k} \right)$

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Theorem (S.M., S. Onn, 2012)

Given a set of non-partitionable integers $W = \{w_1,\ldots,w_n\}$ encoded as a system of polynomial equations as above, there exists a minimum-degree Nullstellensatz certificate for the non-existence of a partition of W as follows:

$$1 = \sum_{i=1}^{n}\Big(\sum_{\substack{k \text{ even} \\ k\le n-1}}\sum_{s\in S_k^{n\setminus i}} c_{i,s}x^s\Big)(x_i^2-1) + \Big(\sum_{\substack{k \text{ odd} \\ k\le n}}\sum_{s\in S_k^n} b_s x^s\Big)\Big(\sum_{i=1}^{n} w_i x_i\Big).$$

Moreover, every Nullstellensatz certificate associated with the above system of polynomial equations contains exactly one monomial for each of the even parity subsets of $S_k^{n\setminus i}$, and exactly one monomial for each of the odd parity subsets of S_k^n.

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Note: certificate is both high degree and dense.

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The Partition Matrix: Extract a Square Linear System

Let $W = \{w_1, w_2, w_3\}$.

$$\begin{bmatrix} w_3 & w_2 & w_1 & 0 \\ w_2 & w_3 & 0 & w_1 \\ w_1 & 0 & w_3 & w_2 \\ 0 & w_1 & w_2 & w_3 \end{bmatrix}$$

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$-$	$+$
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Let $W = \{w_1, w_2, w_3\}$.

$$\begin{bmatrix} w_3 & w_2 & w_1 & 0 \\ w_2 & w_3 & 0 & w_1 \\ w_1 & 0 & w_3 & w_2 \\ 0 & w_1 & w_2 & w_3 \end{bmatrix}$$

$-$	$+$
	$w_1 + w_2 + w_3$
$-w_1$	$+w_2 + w_3$
$-w_2$	$+w_1 \quad + w_3$
$-w_1 - w_2$	$+w_3$

The **determinant** of the above **partition matrix** is the

$$\underbrace{(w_1 + w_2 + w_3)(-w_1 + w_2 + w_3)(w_1 - w_2 + w_3)(-w_1 - w_2 + w_3)}_{\textbf{partition polynomial}}$$

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Let $W = \{w_1, w_2, w_3\}$.

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$-$	$+$
	$w_1 + w_2 + w_3$
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$$\underbrace{(w_1 + w_2 + w_3)(-w_1 + w_2 + w_3)(w_1 - w_2 + w_3)(-w_1 - w_2 + w_3)}_{\textbf{partition polynomial}}$$

Theorem (S.M., S. Onn, D.V. Pasechnik, 2015)

The determinant of the partition matrix is the partition polynomial.

Hilbert's Nullstellensatz *Numeric* Coefficients and the Partition Polynomial

Given a square non-singular matrix A, Cramer's rule states that $Ax = b$ can be solved according to the formula

$$x_i = \frac{\det(A|_b^i)}{\det(A)} \ ,$$

where $A|_b^i$ is the matrix A with the i-th column replaced with the right-hand side vector b.

Recall the non-partitionable $W = \{1, 3, 5, 2\}$:

$$
\begin{aligned}
1 = &\left(-\frac{155}{693} + \frac{842}{3465}x_2x_3 - \frac{188}{693}x_2x_4 + \frac{908}{3465}x_3x_4\right)\mathbf{(x_1^2 - 1)} \\
&+ \left(-\frac{1}{231} + \frac{842}{1155}x_1x_3 - \frac{188}{231}x_1x_4 + \frac{292}{1155}x_3x_4\right)\mathbf{(x_2^2 - 1)} \\
&+ \left(-\frac{467}{693} + \frac{842}{693}x_1x_2 + \frac{908}{693}x_1x_4 + \frac{292}{693}x_2x_4\right)\mathbf{(x_3^2 - 1)} \\
&+ \left(-\frac{68}{693} - \frac{376}{693}x_1x_2 + \frac{1816}{3465}x_1x_3 + \frac{584}{3465}x_2x_3\right)\mathbf{(x_4^2 - 1)} \\
&+ \left(\frac{155}{693}x_1 + \frac{1}{693}x_2 + \frac{467}{3465}x_3 + \frac{34}{693}x_4 - \frac{842}{3465}x_1x_2x_3\right. \\
&\left. + \frac{188}{693}x_1x_2x_4 - \frac{908}{3465}x_1x_3x_4 - \frac{292}{3465}x_2x_3x_4\right)\mathbf{(x_1 + 3x_2 + 5x_3 + 2x_4)}\ .
\end{aligned}
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\begin{aligned}
-51975 = &(1+3+5+2)(-1+3+5+2)(1-3+5+2)(1+3-5+2) \\
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Via Cramer's rule, we see that the unknown b_4 is equal to

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&\left.+ \frac{188}{693}x_1x_2x_4 - \frac{908}{3465}x_1x_3x_4 - \frac{292}{3465}x_2x_3x_4\right)(x_1 + 3x_2 + 5x_3 + 2x_4)\ .
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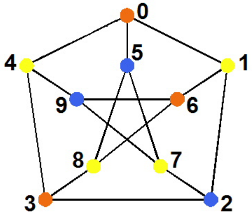
Definition of Graph Coloring Problem

- **Graph coloring:** Given a graph G, and an integer k, can the vertices be colored with k colors in such a way that no two adjacent vertices are the same color?
- **Petersen Graph: 3-colorable**



Graph 3-Coloring as a System of Polynomial Equations over $\mathbb{C}$ (D. Bayer)

- one **variable** per **vertex**: $x_1, \ldots, x_n$

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- **Theorem:** Let G be a graph encoded as the above $(n + m)$ system of equations. Then this system has a solution if and only if G is 3-colorable.

Petersen Graph $\Longrightarrow$ System of Polynomial Equations

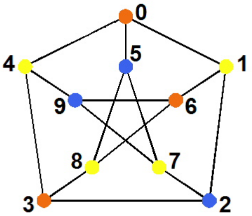

Figure: Is the Petersen graph 3-colorable?

$$
\begin{array}{ll}
x_0^3 - 1 = 0, x_1^3 - 1 = 0, & x_0^2 + x_0x_1 + x_1^2 = 0, x_0^2 + x_0x_4 + x_4^2 = 0 \\
x_2^3 - 1 = 0, x_3^3 - 1 = 0, & x_0^2 + x_0x_5 + x_5^2 = 0, x_1^2 + x_1x_2 + x_2^2 = 0 \\
x_4^3 - 1 = 0, x_5^3 - 1 = 0, & x_1^2 + x_1x_6 + x_6^2 = 0, x_2^2 + x_2x_3 + x_3^2 = 0 \\
x_6^3 - 1 = 0, x_7^3 - 1 = 0, & \quad \cdots\cdots \qquad\qquad \cdots\cdots \\
x_8^3 - 1 = 0, x_9^3 - 1 = 0, & x_6^2 + x_6x_8 + x_8^2 = 0, x_7^2 + x_7x_9 + x_9^2 = 0
\end{array}
$$

Where is the Infinite Family of Graphs that Grow over $\mathbb{C}$?

- Flower, Kneser, Grötzsch, Jin, Mycielski graphs have degree 4.

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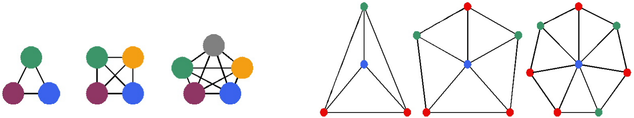
- **Theorem:** Every Nullstellensatz certificate of a non-3-colorable graph has degree *at least* four.
- **Theorem:** For $n \geq 4$, a minimum-degree Nullstellensatz certificate of non-3-colorability for cliques and odd wheels has degree exactly four.



Graph 3-Coloring as a System of Polynomial Equations over $\overline{\mathbb{F}_2}$ (inspired by Bayer)

- one **variable** per **vertex**: $x_1, \ldots, x_n$
- **vertex polynomials:** For every vertex $i = 1, \ldots, n$,

$$x_i^3 + 1 = 0$$

- **edge polynomials:** For every edge $(i, j) \in E(G)$,

$$x_i^2 + x_i x_j + x_j^2 = 0$$

- **Theorem:** Let G be a graph encoded as the above $(n + m)$ system of equations. Then this system has a solution if and only if G is 3-colorable.

Where is the Infinite Family of Graphs that Grow over $\overline{\mathbb{F}_2}$?

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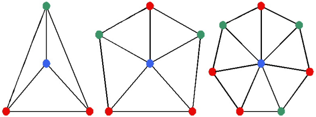

Experimental results for NulLA 3-colorability

Graph	*vertices*	*edges*	*rows*	*cols*	*deg*	*sec*
Mycielski 7	95	755	64,281	71,726		
Mycielski 9	383	7,271	2,477,931	2,784,794		
Mycielski 10	767	22,196	15,270,943	17,024,333		
$(8,3)$-Kneser	56	280	15,737	15,681		
$(10,4)$-Kneser	210	1,575	349,651	330,751		
$(12,5)$-Kneser	792	8,316	7,030,585	6,586,273		
$(13,5)$-Kneser	1,287	36,036	45,980,650	46,378,333		
1-Insertions_5	202	1,227	268,049	247,855		
2-Insertions_5	597	3,936	2,628,805	2,349,793		
3-Insertions_5	1,406	9,695	15,392,209	13,631,171		
ash331GPIA	662	4,185	3,147,007	2,770,471		
ash608GPIA	1,216	7,844	10,904,642	9,538,305		
ash958GPIA	1,916	12,506	27,450,965	23,961,497		

Table: Graphs without 4-cliques.

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$(12,5)$-Kneser	792	8,316	7,030,585	6,586,273	1	466.47
$(13,5)$-Kneser	1,287	36,036	45,980,650	46,378,333	1	216105
1-Insertions_5	202	1,227	268,049	247,855	1	1.69
2-Insertions_5	597	3,936	2,628,805	2,349,793	1	18.23
3-Insertions_5	1,406	9,695	15,392,209	13,631,171	1	83.45
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ash958GPIA	1,916	12,506	27,450,965	23,961,497	1	90.41

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Comparison with Graph Coloring Heuristics

- *A Branch-and-Cut algorithm for graph coloring* by Isabel Méndez-Díaz and Paula Zabala (2006)

			B&C		DSATUR		**NulLA**	
G	n	m	lb	up	lb	up	deg	sec
4-Insertions_3.col	79	156	3	4	2	4		
3-Insertions_4.col	281	1,046	3	5	2	5		
4-Insertions_4.col	475	1,795	3	5	2	5		
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2-Insertions_5.col	597	3,936	3	6	2	6	1	19
3-Insertions_5.col	1,406	9,695	3	6	2	6	1	169

Nullstellensatz Certificates of Non-3-colorability

Theorem (S.M., 2008)

The minimum-degree certificate for non-3-colorability grows as $1, 4, 7, 10, \ldots$.

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  - *Directed Edge Cover* interpretation (De Loera, Hillar, Malkin, Omar, "Recognizing Graph Theoretic Properties with Polynomial Ideals", **2010**)

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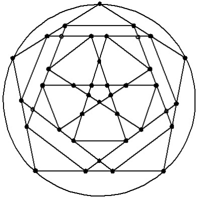
- **Degree Four Certificates:** Open Question!!

What if the Nullstellensatz certificate is *not* degree 1?



degree 4 certificate

$7,585,826 \times 9,887,481$

over 4 hours

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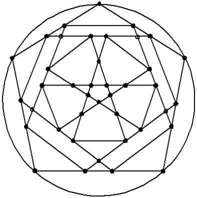

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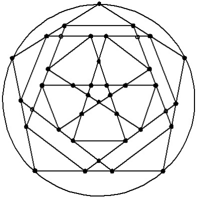

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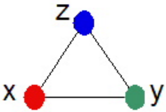

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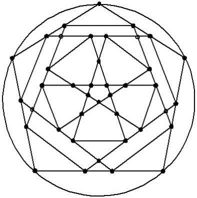

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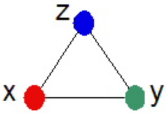

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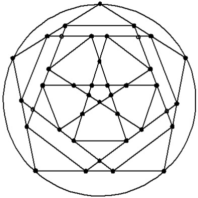

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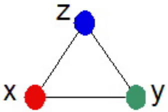

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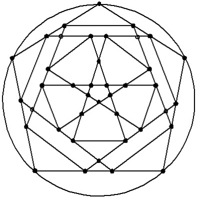

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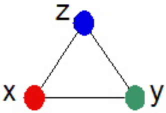

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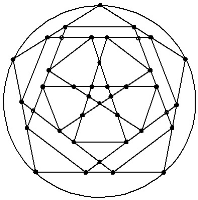

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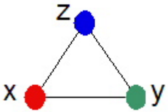

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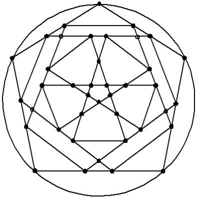

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.2 seconds

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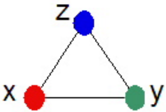

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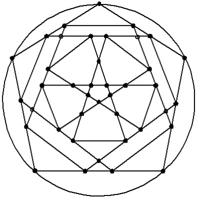

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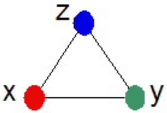

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Appending equations to the system can *reduce* the degree!

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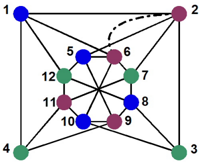

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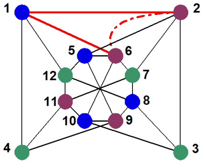

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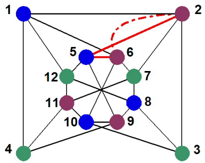

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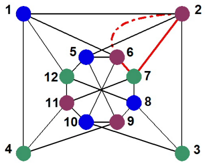

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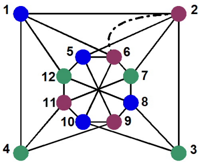

Alternative Nullstellensätze

$$x_1^{\alpha_1} \cdots x_n^{\alpha_n} = \sum_{i=1}^{s} \beta_i f_i$$

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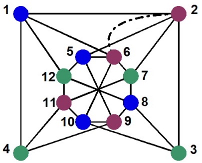

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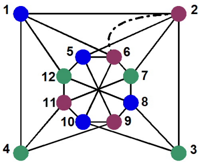

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$$\begin{aligned} x_1x_8x_9 = {} & (x_1 + x_2)(x_1^2 + x_1x_2 + x_2^2) + (x_4 + x_9 + x_{12})(x_1^2 + x_1x_4 + x_4^2) + \cdots + \\ & + (x_1 + x_4 + x_8)(x_1^2 + x_1x_{12} + x_{12}^2) + (x_2 + x_7 + x_8)(x_2^2 + x_2x_3 + x_3^2) \\ & + (x_8 + x_9)\underbrace{(x_1^2 + x_2^2 + x_6^2)}_{\text{triangle equation}} + (x_9)\underbrace{(x_2^2 + x_5^2 + x_6^2)}_{\text{triangle equation}} + (x_8)\underbrace{(x_2^2 + x_6^2 + x_7^2)}_{\text{triangle equation}}. \end{aligned}$$

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$$(i)x_1^3 + (1 - i)x_2 + (3/2)x_3^7 \in \mathbb{C}[x_1, x_2, x_3]$$

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Let $I = \langle f_1, \ldots, f_s \rangle$, $g_1, g_2 \in \mathbb{K}[x_1, \ldots, x_n]$ and $g_1 g_2 \in I$.

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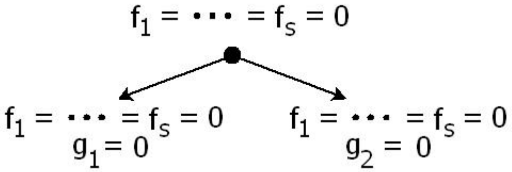

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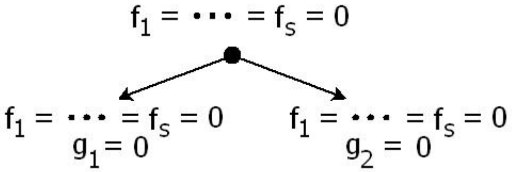

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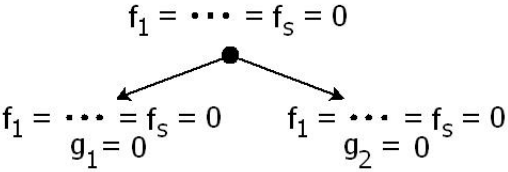

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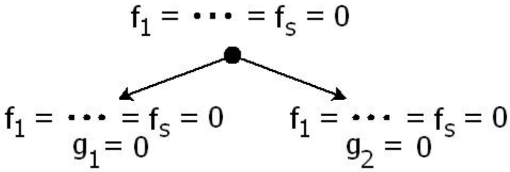

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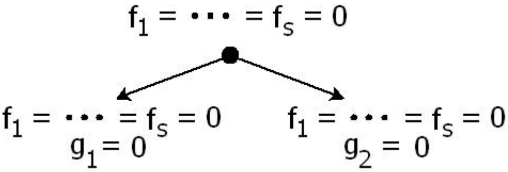

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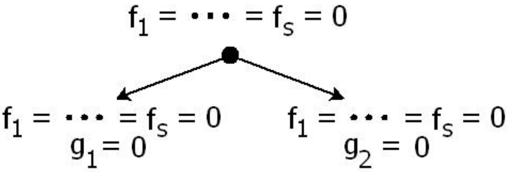

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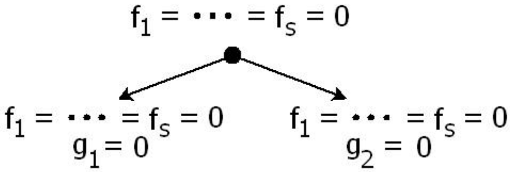

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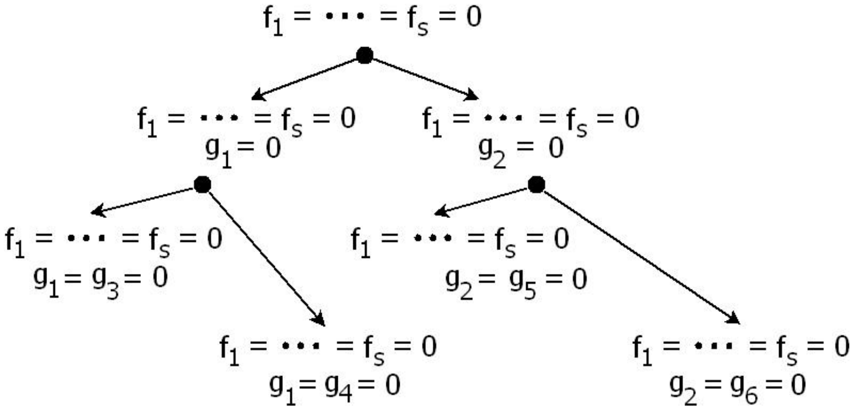

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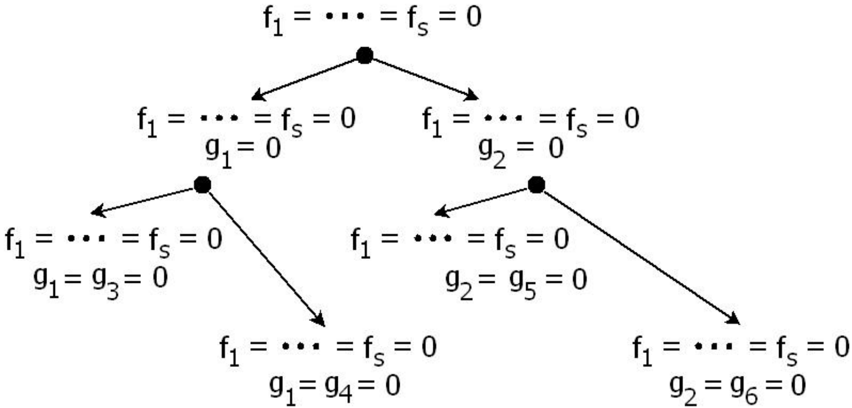

Remark: Over $\mathbb{F}_2$, $(x_i + 1)(x_i^2 + x_i + 1)$

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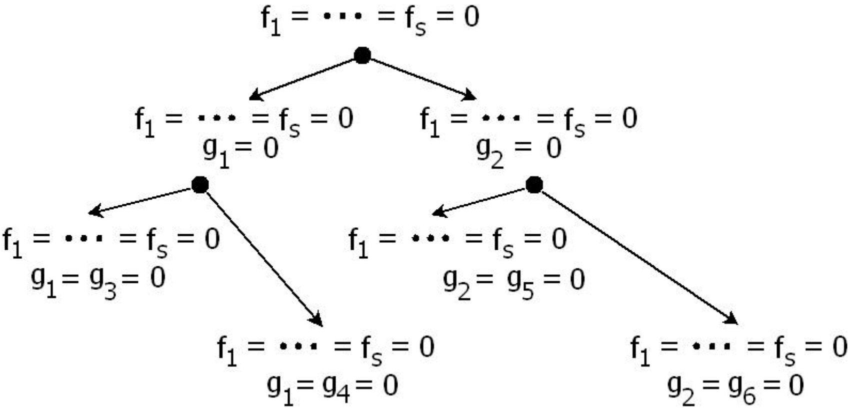
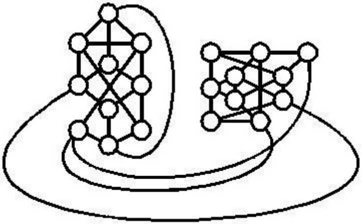


Remark: Over $\mathbb{F}_2$, $(x_i + 1)(x_i^2 + x_i + 1) = x_i^3 + 1$.

Branching Computational Example



degree four certificate
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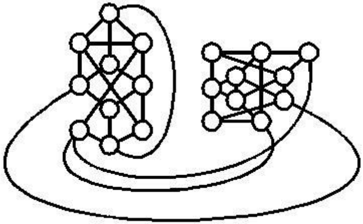

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$\Downarrow$
9 degree one subproblems
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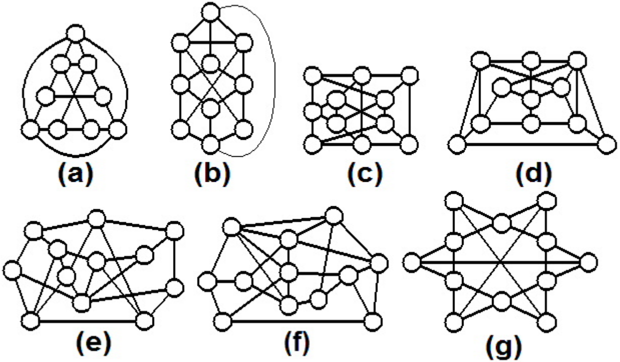
Where is our Infinite Family?



near-4-clique free 4-critical graphs by Nishihara-Mizuno

1. Choose a base graph.
2. Choose another graph.
3. Link using Hajós' join.
4. Repeat.

Hard Instances: Branching vs. Non-Branching

			NulLA w/o branching				**NulLA** w/ branching	
G	n	m	rows	cols	deg	sec	# of subprobs	sec
G_0	10	18	198	181	1	0		
G_1	20	37	178,012	329,916	4	6		
G_2	30	55	1,571,328	2,257,211	4	52		
G_3	39	72	6,481,224	8,072,429	4	201		
G_4	49	90	22,054,196	24,390,486	≥ 7	773		
G_5	60	110	-	-	-	-		
G_6	69	127	-	-	-	-		
G_7	78	144	-	-	-	-		

Table: Hard instances of graph 3-colorability: MUGs

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The minimum-degree certificate for graph 3-colorability grows as $1, 4, 7, 10, \ldots$.

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Open Question

What is the relationship between the number of vertices in the graph, and this growth pattern?

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G_2	30	55	1,571,328	2,257,211	4	52	83	.31
G_3	39	72	6,481,224	8,072,429	4	201	479	2.86
G_4	49	90	22,054,196	24,390,486	≥ 7	773	6,131	53.48
G_5	60	110	-	-	-	-	67,163	946.66
G_6	69	127	-	-	-	-	103,787	2031.98
G_7	78	144	-	-	-	-	297,371	7058.14

Table: Hard instances of graph 3-colorability: MUGs

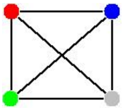
Theorem

The minimum-degree certificate for graph 3-colorability grows as $1, 4, 7, 10, \ldots$.

Open Question

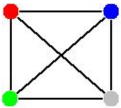
What is the relationship between the number of vertices in the graph, and this growth pattern?

Using Symmetry to Reduce the Size of the Linear System



Consider the complete graph K_4.

Using Symmetry to Reduce the Size of the Linear System



Consider the complete graph K_4. A degree-one Hilbert Nullstellensatz certificate for non-3-colorability, over $\overline{\mathbb{F}_2}$ is

$$
\begin{aligned}
1 = {} & c_0(x_1^3 + 1) \\
& + (c_{12}^1 x_1 + c_{12}^2 x_2 + c_{12}^3 x_3 + c_{12}^4 x_4)(x_1^2 + x_1x_2 + x_2^2) \\
& + (c_{13}^1 x_1 + c_{13}^2 x_2 + c_{13}^3 x_3 + c_{13}^4 x_4)(x_1^2 + x_1x_3 + x_3^2) \\
& + (c_{14}^1 x_1 + c_{14}^2 x_2 + c_{14}^3 x_3 + c_{14}^4 x_4)(x_1^2 + x_1x_4 + x_4^2) \\
& + (c_{23}^1 x_1 + c_{23}^2 x_2 + c_{23}^3 x_3 + c_{23}^4 x_4)(x_2^2 + x_2x_3 + x_3^2) \\
& + (c_{24}^1 x_1 + c_{24}^2 x_2 + c_{24}^3 x_3 + c_{24}^4 x_4)(x_2^2 + x_2x_4 + x_4^2) \\
& + (c_{34}^1 x_1 + c_{34}^2 x_2 + c_{34}^3 x_3 + c_{34}^4 x_4)(x_3^2 + x_3x_4 + x_4^2)
\end{aligned}
$$

Matrix associated with K_4 Nullstellensatz Certificate: $M_{F,1}$

	c_0	c_{12}^1	c_{12}^2	c_{12}^3	c_{12}^4	c_{13}^1	c_{13}^2	c_{13}^3	c_{13}^4	c_{14}^1	c_{14}^2	c_{14}^3	c_{14}^4	c_{23}^1	c_{23}^2	c_{23}^3	c_{23}^4	c_{24}^1	c_{24}^2	c_{24}^3	c_{24}^4	c_{34}^1	c_{34}^2	c_{34}^3	c_{34}^4
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x_1^3	1	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$x_1^2x_2$	0	1	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$x_1^2x_3$	0	0	0	1	0	1	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
$x_1^2x_4$	0	0	0	0	1	0	0	0	1	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
$x_1x_2^2$	0	1	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0
$x_1x_2x_3$	0	0	0	1	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
$x_1x_2x_4$	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0
$x_1x_3^2$	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0
$x_1x_3x_4$	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0
$x_1x_4^2$	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	1	0	0	0	1	0	0	0
x_2^3	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0
$x_2^2x_3$	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	1	0	0	0	0	0
$x_2^2x_4$	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	0	0	0
$x_2x_3^2$	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	1	0	0
$x_2x_3x_4$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	1	0	0
$x_2x_4^2$	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	1	0	1	0	0
x_3^3	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0
$x_3^2x_4$	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	1
$x_3x_4^2$	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	1	1
x_4^3	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	1

Using Symmetry to Reduce the Size of the Linear System

Suppose a finite permutation group G acts on the variables $x_1, \ldots, x_n$.

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Suppose a finite permutation group G acts on the variables $x_1, \ldots, x_n$. Assume that the set F of polynomials is invariant under the action of G, i.e., $g(f_i) \in F$ for each $f_i \in F$.

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Suppose a finite permutation group G acts on the variables $x_1, \ldots, x_n$. Assume that the set F of polynomials is invariant under the action of G, i.e., $g(f_i) \in F$ for each $f_i \in F$.

We will use this group to reduce the size of the matrix.

Matrix associated with K_4 Nullstellensatz Certificate: $M_{F,1}$

	c_0	c_{12}^1	c_{13}^1	c_{14}^1	c_{12}^2	c_{13}^3	c_{14}^4	c_{12}^3	c_{13}^4	c_{14}^2	c_{12}^4	c_{13}^2	c_{14}^3	c_{23}^1	c_{34}^1	c_{24}^1	c_{23}^2	c_{34}^3	c_{24}^4	c_{24}^2	c_{23}^3	c_{34}^4	c_{34}^2	c_{24}^3	c_{23}^4
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x_1^3	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$x_1^2x_2$	0	1	0	0	1	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
$x_1^2x_3$	0	0	1	0	0	1	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
$x_1^2x_4$	0	0	0	1	0	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$x_1x_2^2$	0	1	0	0	1	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0
$x_1x_3^2$	0	0	1	0	0	1	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0
$x_1x_4^2$	0	0	0	1	0	0	1	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0
$x_1x_2x_3$	0	0	0	0	0	0	0	1	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0
$x_1x_2x_4$	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0
$x_1x_3x_4$	0	0	0	0	0	0	0	0	1	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0
x_2^3	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0
x_3^3	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0
x_4^3	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0
$x_2^2x_3$	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	1	0
$x_3^2x_4$	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	1
$x_2x_4^2$	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	1	0	0	1	0	0
$x_2^2x_4$	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	1	0	0	0	0	1
$x_2x_3^2$	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	1	0	1	0	0
$x_3x_4^2$	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	1	0	1	0
$x_2x_3x_4$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1

Action of Z_3 by $(2, 3, 4)$: each row block represents an orbit.

Matrix associated with K_4 Nullstellensatz Certificate: $M_{F,1,G}$

	$\bar{c}_0$	$\bar{c}_{12}^1$	$\bar{c}_{12}^2$	$\bar{c}_{12}^3$	$\bar{c}_{12}^4$	$\bar{c}_{23}^1$	$\bar{c}_{23}^2$	$\bar{c}_{24}^2$	$\bar{c}_{34}^2$
$Orb(1)$	1	0	0	0	0	0	0	0	0
$Orb(x_1^3)$	1	**3**	0	0	0	0	0	0	0
$Orb(x_1^2x_2)$	0	1	1	1	1	0	0	0	0
$Orb(x_1x_2^2)$	0	1	1	0	0	**2**	0	0	0
$Orb(x_1x_2x_3)$	0	0	0	1	1	1	0	0	0
$Orb(x_2^3)$	0	0	1	0	0	0	1	1	0
$Orb(x_2^2x_3)$	0	0	0	1	0	0	1	1	1
$Orb(x_2^2x_4)$	0	0	0	0	1	0	1	1	1
$Orb(x_2x_3x_4)$	0	0	0	0	0	0	0	0	**3**

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	$\bar{c}_0$	$\bar{c}_{12}^1$	$\bar{c}_{12}^2$	$\bar{c}_{12}^3$	$\bar{c}_{12}^4$	$\bar{c}_{23}^1$	$\bar{c}_{23}^2$	$\bar{c}_{24}^2$	$\bar{c}_{34}^2$
$Orb(1)$	1	0	0	0	0	0	0	0	0
$Orb(x_1^3)$	1	**3**	0	0	0	0	0	0	0
$Orb(x_1^2x_2)$	0	1	1	1	1	0	0	0	0
$Orb(x_1x_2^2)$	0	1	1	0	0	**2**	0	0	0
$Orb(x_1x_2x_3)$	0	0	0	1	1	1	0	0	0
$Orb(x_2^3)$	0	0	1	0	0	0	1	1	0
$Orb(x_2^2x_3)$	0	0	0	1	0	0	1	1	1
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$Orb(x_2x_3x_4)$	0	0	0	0	0	0	0	0	**3**

$\overset{(\text{mod } 2)}{\equiv}$

	$\bar{c}_0$	$\bar{c}_{12}^1$	$\bar{c}_{12}^2$	$\bar{c}_{12}^3$	$\bar{c}_{12}^4$	$\bar{c}_{23}^1$	$\bar{c}_{23}^2$	$\bar{c}_{24}^2$	$\bar{c}_{34}^2$
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$Orb(x_1^3)$	1	1	0	0	0	0	0	0	0
$Orb(x_1^2x_2)$	0	1	1	1	1	0	0	0	0
$Orb(x_1x_2^2)$	0	1	1	0	0	0	0	0	0
$Orb(x_1x_2x_3)$	0	0	0	1	1	1	0	0	0
$Orb(x_2^3)$	0	0	1	0	0	0	1	1	0
$Orb(x_2^2x_3)$	0	0	0	1	0	0	1	1	1
$Orb(x_2^2x_4)$	0	0	0	0	1	0	1	1	1
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Solution to Orbit Matrix Proves Certificate Existence

- **Theorem:** Let $\mathbb{K}$ be an algebraically-closed field. Let $F = \{f_1, \ldots, f_s\} \subseteq \mathbb{K}[x_1, \ldots, x_n]$ and suppose F is closed under the action of the group G on the variables. Suppose that the order of the group $|G|$ and the characteristic of the field $\mathbb{K}$ are relatively prime. Then, the degree d Nullstellensatz linear system of equations $M_{F,d}\, y = b_{F,d}$ has a solution over $\mathbb{K}$ if and only if the system of linear equations $\overline{M}_{F,d,G}\overline{y} = \overline{b}_{F,d,G}$ has a solution over $\mathbb{K}$.

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In other words, if the orbit matrix has a solution, so does the original matrix.

To Summarize

- For the **Independent Set** and **Partition** problem, the certificates are generally both high degree and dense.
- For the **Graph-3-coloring** problem, the certificates are surprisingly generally of low degree and sparse (with the exception of the Nishihara-Mizuno graphs!)
- The Nullstellensatz certificate of infeasibility helped identify a new class of graph-3-coloring problems solvable in polynomial-time!
- There are lots of computational methods for exploiting the structure of a Nullstellensatz-based algorithm: adding special polynomials, searching for alternative-form certificates, branching, and using symmetry.

Nullstellensatz Certificates for Problems in P

Question

Given a combinatorial problem in P, does there **exist** an encoding such that the Nullstellensatz certificates have polynomial size?

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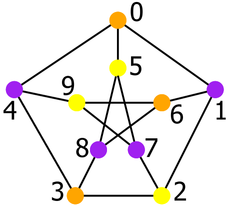

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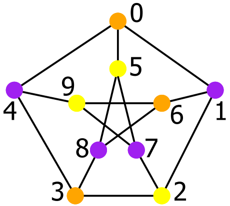

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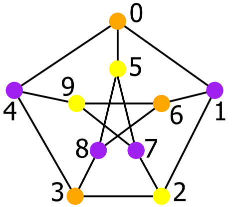

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A graph G is not-2-colorable $\iff$ G contains an odd cycle.

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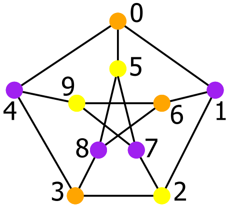

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A graph G is not-2-colorable $\iff G$ contains an odd cycle.

- $(x_i^2 - 1) = 0 \ , \forall i \in V(G)$ and $(x_i + x_j) = 0 \ , \forall (i,j) \in E(G)$ $(\mathbb{C})$

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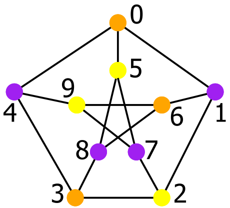

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$$-(x_0^2 - 1)$$

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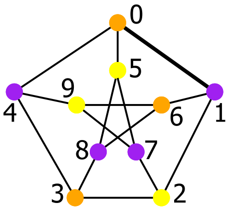

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$$-(x_0^2 - 1) + \frac{1}{2}x_0(x_0 + x_1)$$

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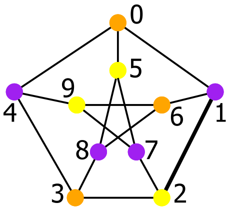

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$$-(x_0^2 - 1) + \frac{1}{2}x_0(x_0 + x_1) - \frac{1}{2}x_0(x_1 + x_2)$$

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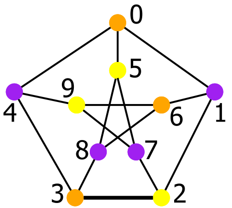

Fact

A graph G is not-2-colorable $\Longleftrightarrow$ G contains an odd cycle.

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$$-(x_0^2 - 1) + \frac{1}{2}x_0(x_0 + x_1) - \frac{1}{2}x_0(x_1 + x_2) + \frac{1}{2}x_0(x_2 + x_3)$$

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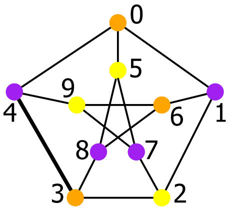

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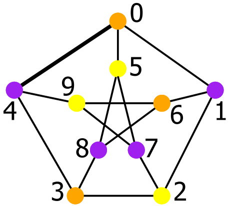
$$-(x_0^2 - 1) + \frac{1}{2}x_0(x_0 + x_1) - \frac{1}{2}x_0(x_1 + x_2) + \frac{1}{2}x_0(x_2 + x_3) - \frac{1}{2}x_0(x_3 + x_4)$$

Nullstellensatz Certificates for Problems in P

Question

Given a combinatorial problem in P, does there **exist** an encoding such that the Nullstellensatz certificates have polynomial size?

- **Petersen Graph: 3-colorable, not-2-colorable**



Fact

A graph G is not-2-colorable $\iff$ G contains an odd cycle.

- $(x_i^2 - 1) = 0 \ , \forall i \in V(G)$ and $(x_i + x_j) = 0 \ , \forall (i,j) \in E(G)$ $(\mathbb{C})$

$$-(x_0^2 - 1) + \frac{1}{2}x_0(x_0 + x_1) - \frac{1}{2}x_0(x_1 + x_2) + \frac{1}{2}x_0(x_2 + x_3) - \frac{1}{2}x_0(x_3 + x_4) + \frac{1}{2}x_0(x_4 + x_0)$$

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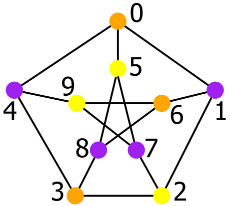

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$$1 = \ -(x_0^2 - 1) + \frac{1}{2}x_0(x_0 + x_1) - \frac{1}{2}x_0(x_1 + x_2) + \frac{1}{2}x_0(x_2 + x_3) - \frac{1}{2}x_0(x_3 + x_4) + \frac{1}{2}x_0(x_4 + x_0)$$

Perfect Matching: Definition and Example

- **Perfect Matching:** A graph G has a perfect matching if there **exists** a set of **matched** edges such that every vertex is incident on a **matched** edge.

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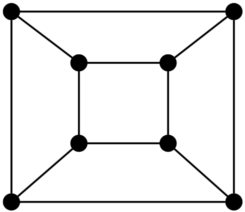

Perfect Matching: Definition and Example

- **Perfect Matching:** A graph G has a perfect matching if there **exists** a set of **matched** edges such that every vertex is incident on a **matched** edge.
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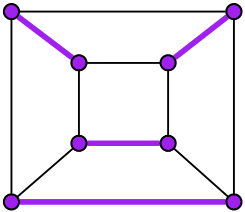

Perfect Matching as a System of Polynomial Equations

- **Proposition:** A graph G has a perfect matching if and only if the following system of polynomial equations over $\mathbb{C}$ has a solution.

$$\sum_{j \in N(i)} x_{ij} + 1 = 0 \qquad \forall i \in V(G)$$

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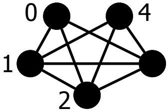

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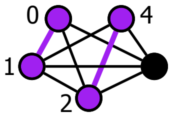

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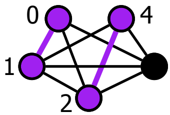
$$\begin{aligned}
1 = {} & (-\tfrac{2}{5}x_{12} - \tfrac{2}{5}x_{13} - \tfrac{2}{5}x_{14} - \tfrac{2}{5}x_{23} - \tfrac{2}{5}x_{24} - \tfrac{2}{5}x_{34} - \tfrac{1}{5})(-1 + x_{01} + x_{02} + x_{03}) \\
& + (-\tfrac{4}{5}x_{02} - \tfrac{4}{5}x_{03} + 2x_{23} - \tfrac{1}{5})(-1 + x_{01} + x_{12} + x_{13} + x_{14}) \\
& + (-\tfrac{4}{5}x_{01} - \tfrac{4}{5}x_{03} + 2x_{13} - \tfrac{1}{5})(-1 + x_{02} + x_{12} + x_{23} + x_{24}) \\
& + (-\tfrac{4}{5}x_{01} - \tfrac{4}{5}x_{02} + 2x_{12} - \tfrac{1}{5})(-1 + x_{03} + x_{13} + x_{23} + x_{34}) \\
& + (\tfrac{6}{5}x_{01} + \tfrac{6}{5}x_{02} + \tfrac{6}{5}x_{03} - 2x_{12} - 2x_{13} - 2x_{23} - \tfrac{1}{5})(-1 + x_{14} + x_{24} + x_{34}) \\
& + \tfrac{8}{5}x_{01}x_{02} + \tfrac{8}{5}x_{01}x_{03} + \tfrac{6}{5}x_{01}x_{12} + \tfrac{6}{5}x_{01}x_{13} - \tfrac{4}{5}x_{01}x_{14} + \tfrac{8}{5}x_{02}x_{03} + \tfrac{6}{5}x_{02}x_{12} \\
& + \tfrac{6}{5}x_{03}x_{13} + \tfrac{6}{5}x_{03}x_{23} - \tfrac{4}{5}x_{03}x_{34} - 4x_{12}x_{13} + 2x_{12}x_{14} - 4x_{12}x_{23} + 2x_{13}x_{14} - \\
& + 2x_{23}x_{24} + 2x_{23}x_{34} + 2x_{12}x_{24};
\end{aligned}$$



Perfect Matching as a System of Polynomial Equations

- **Proposition:** A graph G has a perfect matching if and only if the following system of polynomial equations over $\mathbb{F}_2$ has a solution.

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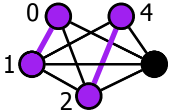

$$\begin{aligned}
1 = & \left(-\frac{2}{5}x_{12} - \frac{2}{5}x_{13} - \frac{2}{5}x_{14} - \frac{2}{5}x_{23} - \frac{2}{5}x_{24} - \frac{2}{5}x_{34} - \frac{1}{5}\right)(-1 + x_{01} + x_{02} + x_{03}) \\
& + \left(-\frac{4}{5}x_{02} - \frac{4}{5}x_{03} + 2x_{23} - \frac{1}{5}\right)(-1 + x_{01} + x_{12} + x_{13} + x_{14}) \\
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& + \left(\frac{6}{5}x_{01} + \frac{6}{5}x_{02} + \frac{6}{5}x_{03} - 2x_{12} - 2x_{13} - 2x_{23} - \frac{1}{5}\right)(-1 + x_{14} + x_{24} + x_{34}) \\
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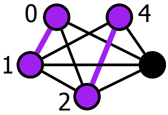

$$\begin{aligned} 1 = &(x_{01} + x_{02} + x_{03} + 1) + (x_{01} + x_{12} + x_{13} + 1) \\ &+ (x_{02} + x_{12} + x_{23} + x_{24} + 1) \\ &+ (x_{03} + x_{13} + x_{23} + x_{34} + 1) \\ &+ (x_{24} + x_{34} + 1) \mod 2 \end{aligned}$$

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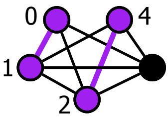

$$\begin{aligned} 1 = & (x_{01} + x_{02} + x_{03} + 1) + (x_{01} + x_{12} + x_{13} + 1) \\ & + (x_{02} + x_{12} + x_{23} + x_{24} + 1) \\ & + (x_{03} + x_{13} + x_{23} + x_{34} + 1) \\ & + (x_{24} + x_{34} + 1) \mod 2 \end{aligned}$$

- **Theorem:** If a graph G has an odd number of vertices, there exists a **degree zero** Nullstellensatz certificate.

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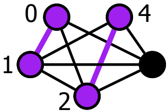

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- **Theorem:** If a graph G has an odd number of vertices, there exists a **degree zero** Nullstellensatz certificate.
- **Question:** What about graphs with an even number of vertices?

Perfect Matching and Bipartite Graphs

Theorem (Hall (1935))

A bipartite graph $G\big(V(A,B),E\big)$ has a perfect matching if and only if for every $U \subseteq V[A]$, $|U| \leq |N(U)|$.

$\|V_R\|$	$\|U\|$	$\|E\|$	deg	$\|V_R\|$	$\|U\|$	$\|E\|$	deg
4	2	10		7	3	34	
4	3	10		7	4	33	
5	2	17		7	5	34	
5	3	16		7	6	37	
5	4	17		8	2	50	
6	2	26		8	3	46	
6	3	24		8	4	44	
6	4	24		8	5	44	
6	5	26		8	6	46	
7	2	37		8	7	50	
7	3	34		9	4	57	
7	4	33		9	5	56	

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$\lvert V_R \rvert$	$\lvert U \rvert$	$\lvert E \rvert$	deg	$\lvert V_R \rvert$	$\lvert U \rvert$	$\lvert E \rvert$	deg
4	2	10	1	7	3	34	1
4	3	10	1	7	4	33	1
5	2	17	1	7	5	34	1
5	3	16	1	7	6	37	1
5	4	17	1	8	2	50	1
6	2	26	1	8	3	46	1
6	3	24	1	8	4	44	
6	4	24	1	8	5	44	
6	5	26	1	8	6	46	1
7	2	37	1	8	7	50	1
7	3	34	1	9	4	57	1
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5	4	17	1	8	2	50	1
6	2	26	1	8	3	46	1
6	3	24	1	8	4	44	2
6	4	24	1	8	5	44	2
6	5	26	1	8	6	46	1
7	2	37	1	8	7	50	1
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7	4	33	2	9	5	56	1

References

http://www.usna.edu/Users/math/margulies

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Thank you for your attention!
Questions and **comments** are most welcome!