GRÖBNER BASES AND APPLICATIONS

Manuel Kauers · Institute for Algebra · JKU

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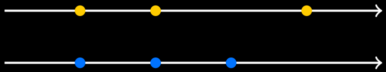

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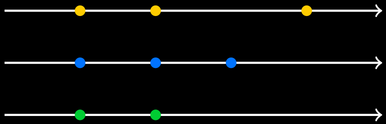

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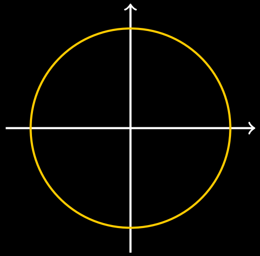

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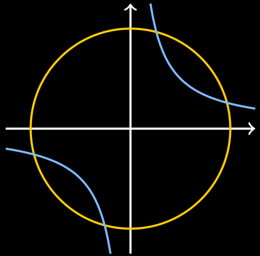

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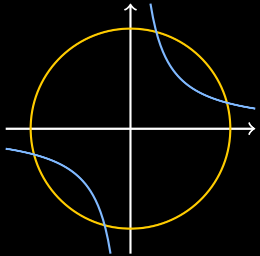

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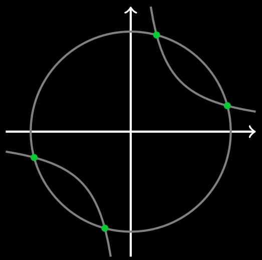

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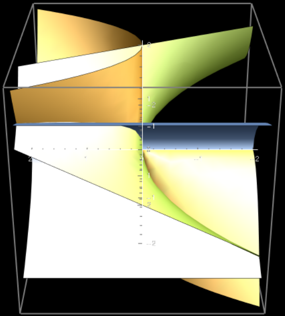

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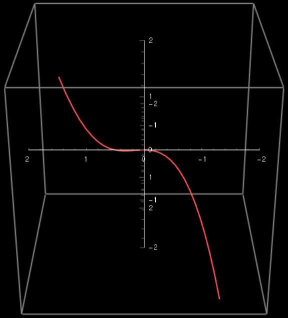

Curves and finite sets of points can be viewed as intersections of such surfaces.

Let K be a field (e.g., $\mathsf{K} = \mathbb{Q}$), and let $\mathsf{K}[X] = \mathsf{K}[x_1, \ldots, x_n]$ be the set of polynomials in $x_1, \ldots, x_n$ with coefficients in K.

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Polynomial equations have implications:

1. $u = 0$ and $v = 0 \Rightarrow u + v = 0$
2. $u = 0$ and v arbitrary $\Rightarrow uv = 0$.

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Among all the bases of a given ideal, the Gröbner basis is one that satisfies a certain minimality condition.

For $n > 1$, divisibility on the set of monomials $x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n}$ is no longer a total ordering, e.g., x^2y and xy^2 are not comparable.

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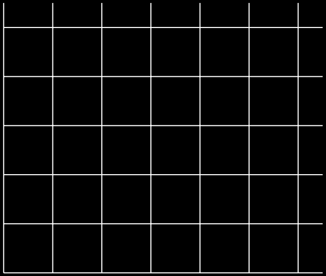
Among all the bases of an ideal, the Gröbner basis is such that the leading terms of its elements are as small as possible.

If a basis of an ideal has a polynomial with head h, then every multiple of h is the head of some element of I.



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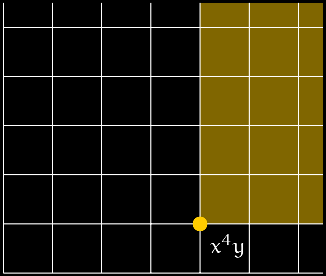

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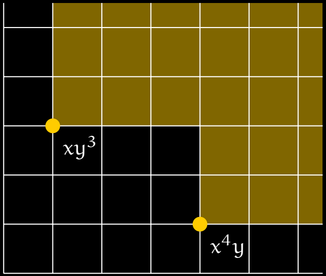

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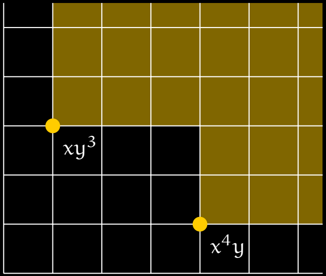

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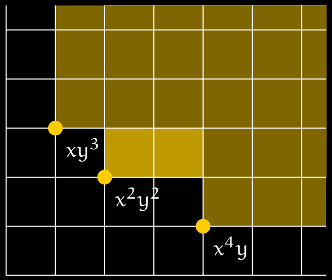

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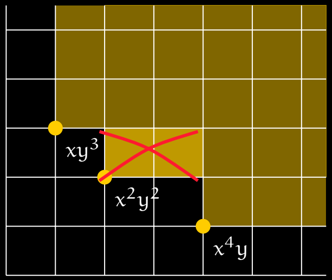

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The basis is called a Gröbner basis if this does not happen.

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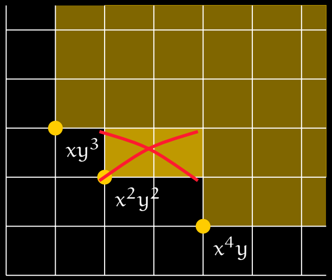

$\{g_1, \ldots, g_k\}$ is a Gröbner basis
$\iff \forall\, p \in \langle g_1, \ldots, g_k \rangle \setminus \{0\}\ \exists\, i \in \{1, \ldots, k\} : \mathrm{Head}(g_i) \mid \mathrm{Head}(p).$

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Fact 4: The computation of a Gröbner basis is a hard problem

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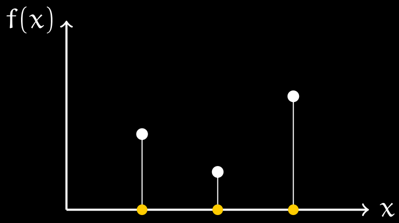

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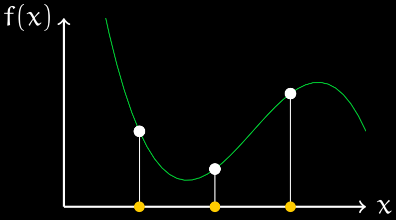

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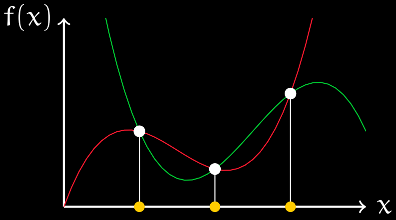

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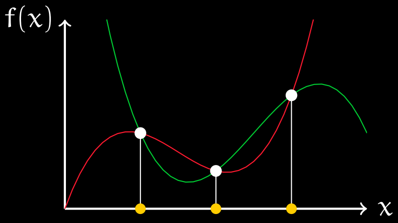

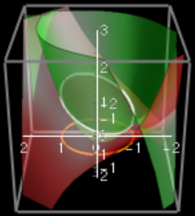

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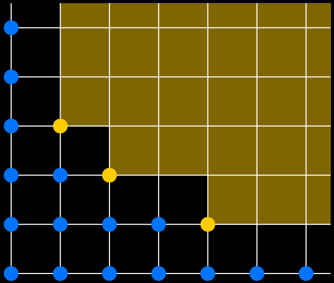
The ideal basis is a Gröbner basis iff each equivalence class contains exactly one polynomial with only blue terms.



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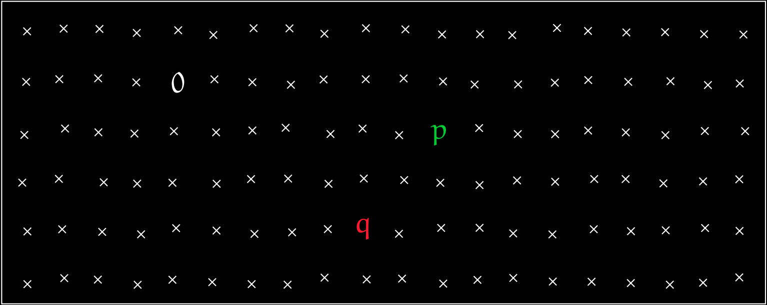

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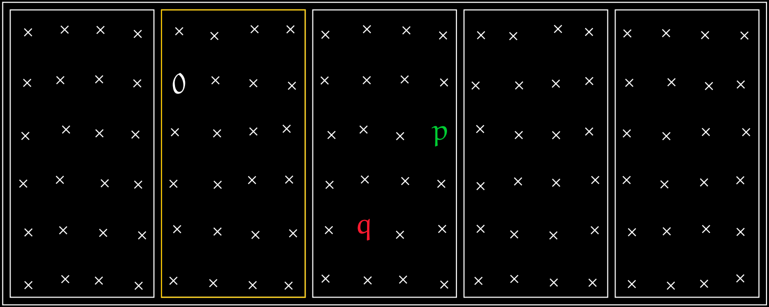

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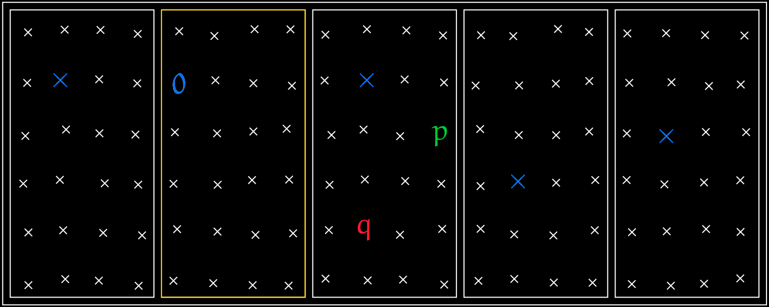

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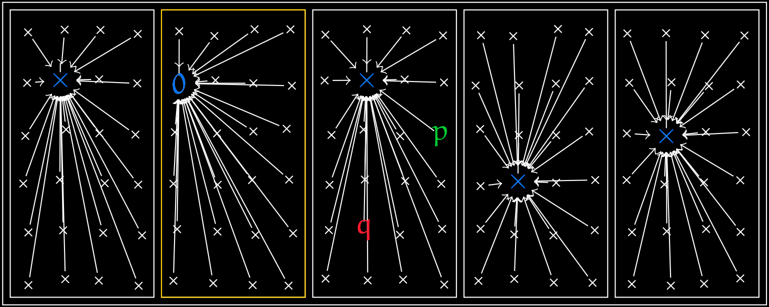

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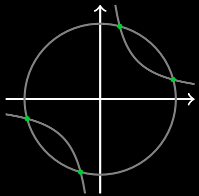

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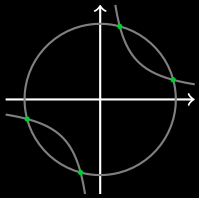

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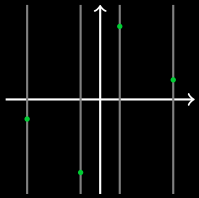

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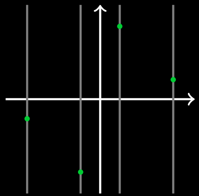

Fact 7*: If G is a Gröbner basis of I, then $G \cap K[X_*]$ is a Gröbner basis of $I \cap K[X_*]$, where $X_* \subseteq \{x_1, \ldots, x_n\}$

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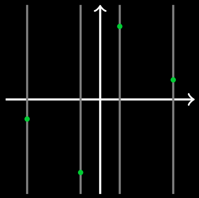

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In particular, we can "triangularize" (and thus solve) a system of polynomial equations.

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Answer: If $p(\alpha) = q(\beta) = 0$, then we can take a basis element of

$$\langle p(x), q(y), z - (x + y) \rangle \cap \mathbb{Q}[z].$$

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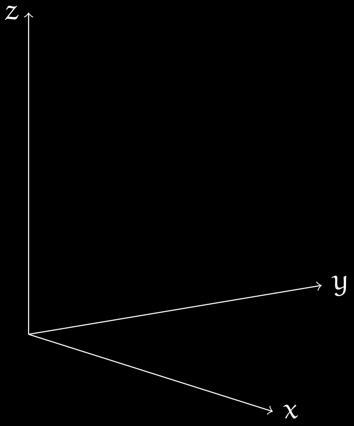

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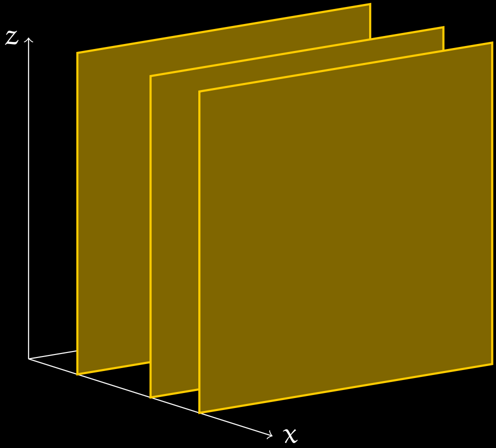

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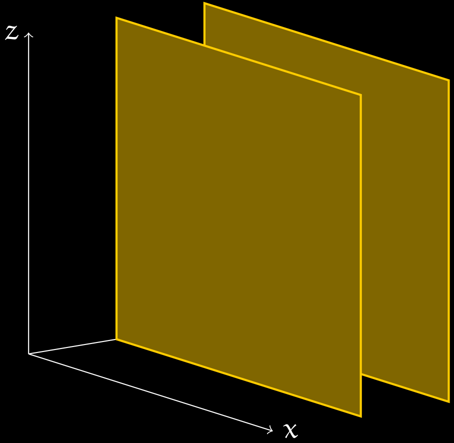

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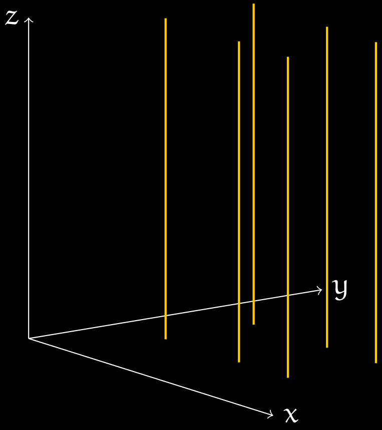

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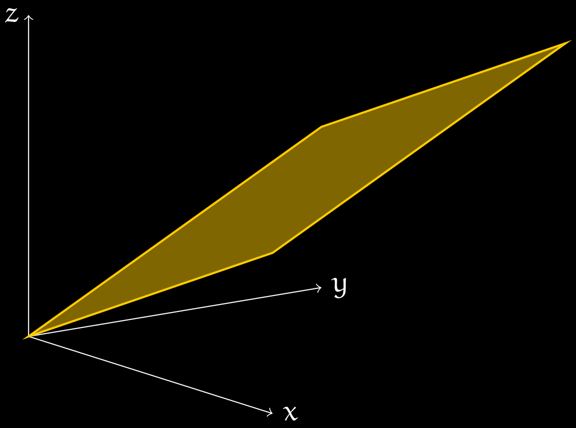

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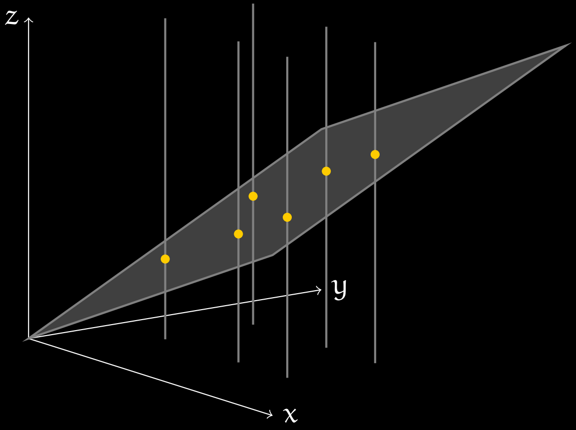

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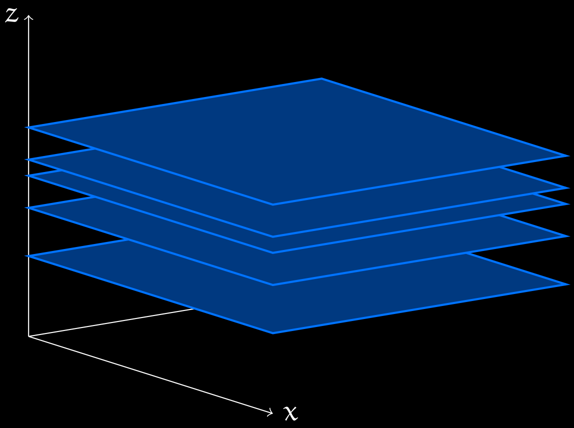

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Answer: Suitable α, β exist if and only if $3b^2 = 4ac$.

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Fact 8:

- If the result is free of x and y, it is a suitable p
- If the result still contains x and y, no suitable p exists.

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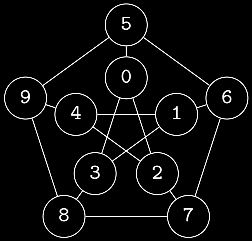

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- Note also: $A \neq 0 \wedge B \neq 0 \iff AB \neq 0$

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The colorings correspond to the solutions of the equation system.

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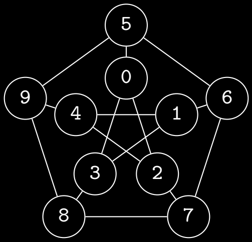
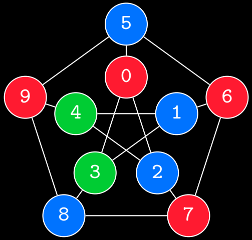


Example: Is the following graph 3-colorable? Yes!



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Choose a term order which eliminates x_1 and minimizes total degree for the remaining variables.

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Choose a term order which eliminates x_1 and minimizes total degree for the remaining variables.

Then $\mathrm{red}(x_1^{123}, I)$ is a monomial which tells us what to buy. If it contains x_1, there is no way.

Example: If there are Chicken McNuggets of sizes 4, 6, 9, and 20, is there a way to buy exactly 123 nuggets?

If so, what is the minimal number of boxes we have to buy?

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For the example, we find $\mathrm{red}(x_1^{123}, I) = x_4^2 x_6 x_9 x_{20}^5$

Some applications of Gröbner Bases

- Computing with Algebraic Numbers
- Quantifier Elimination
- Subring Membership
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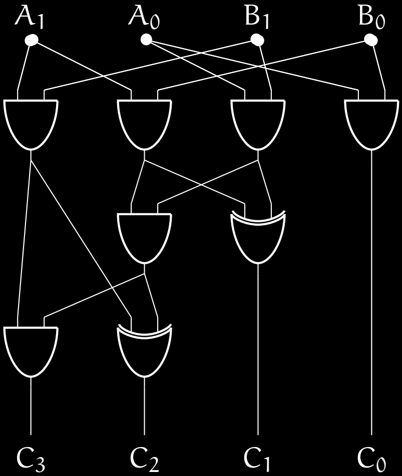
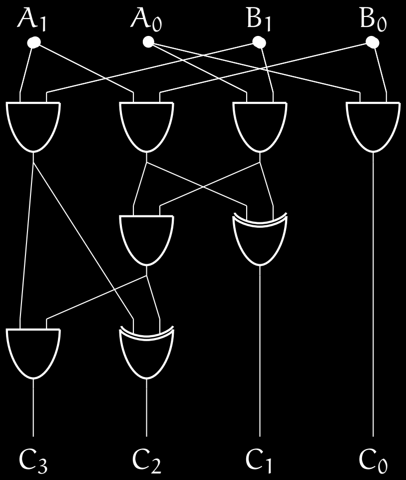
$$10001011001 \times 11011000001$$

$$1\,0\,0\,0\,1\,0\,1\,1\,0\,0\,1 \times 1\,1\,0\,1\,1\,0\,0\,0\,0\,0\,1$$



A_1
A_0
B_1
B_0
C_3
C_2
C_1
C_0


A_1
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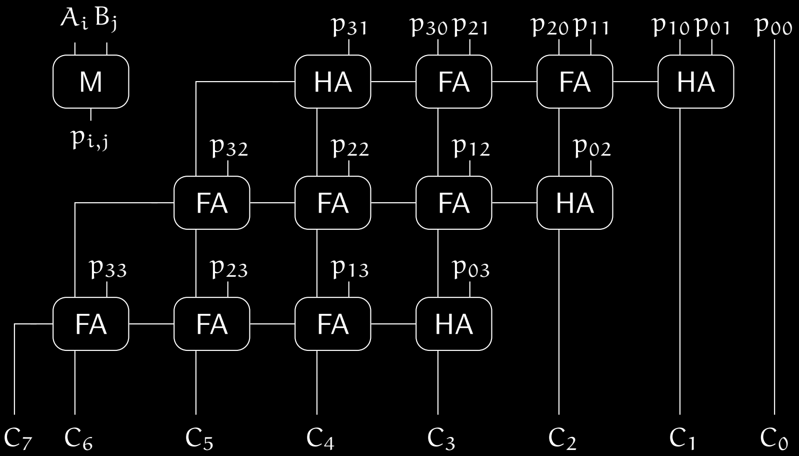
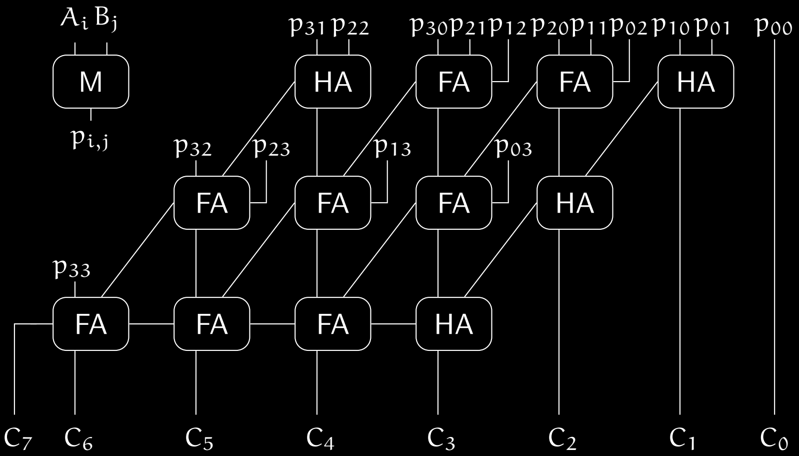


A_i B_j
M
$p_{i,j}$
p_{31}
p_{30} p_{21}
p_{20} p_{11}
p_{10} p_{01}
p_{00}
HA
FA
FA
HA
p_{32}
p_{22}
p_{12}
p_{02}
FA
FA
FA
HA
p_{33}
p_{23}
p_{13}
p_{03}
FA
FA
FA
HA
C_7
C_6
C_5
C_4
C_3
C_2
C_1
C_0



A_i B_j
M
$p_{i,j}$
p_{31} p_{22}
p_{30} p_{21} p_{12} p_{20} p_{11} p_{02} p_{10} p_{01} p_{00}
HA
FA
FA
HA
p_{32} p_{23} p_{13} p_{03}
FA
FA
FA
HA
p_{33}
FA
FA
FA
HA
C_7 C_6 C_5 C_4 C_3 C_2 C_1 C_0

- Every circuit implements a certain function $\{0,1\}^n \to \{0,1\}^m$

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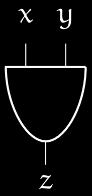

$$z = xy$$

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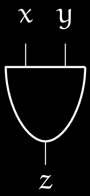
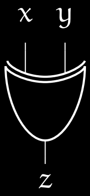


$z = xy$



$z = x + y - 2xy$

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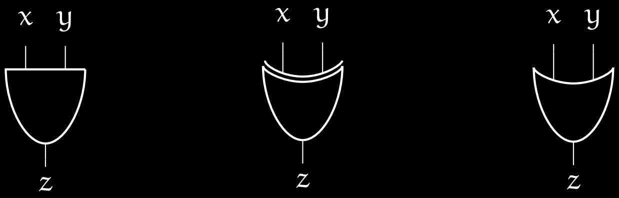

$z = xy$ $\qquad$ $z = x + y - 2xy$ $\qquad$ $z = x + y - xy$

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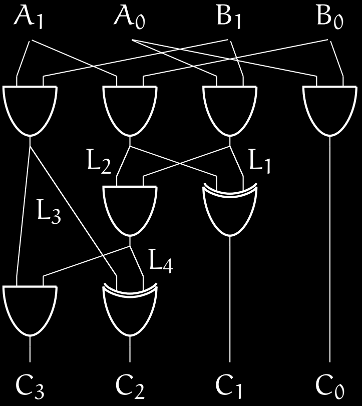

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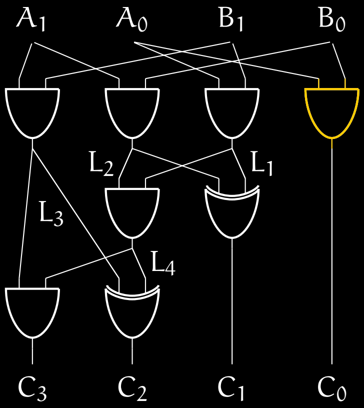

  - $C_0 = A_0 B_0$

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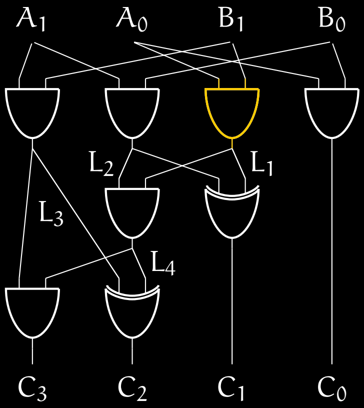

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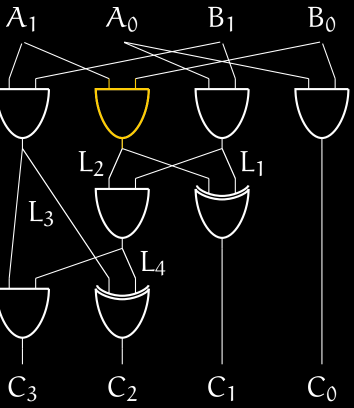

  - $C_0 = A_0 B_0$
  - $L_1 = A_0 B_1$
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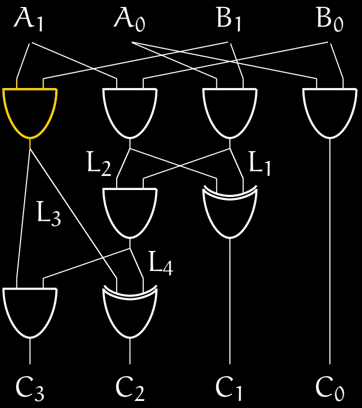

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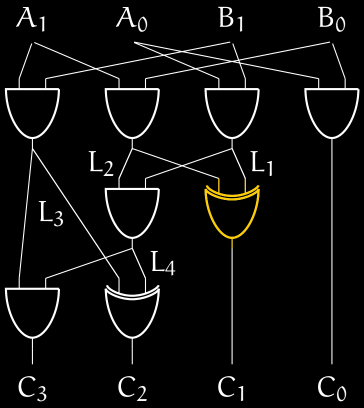

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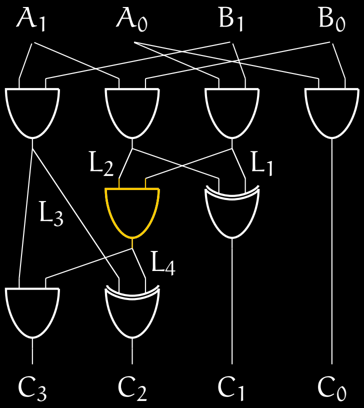

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  - $L_4 = L_1 L_2$

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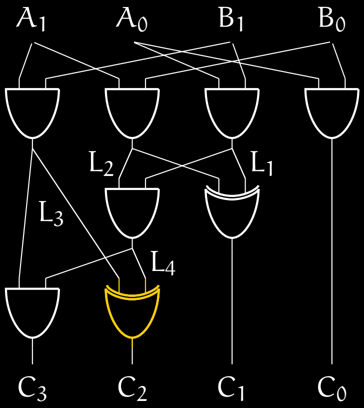

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  - $L_2 = A_1 B_0$
  - $L_3 = A_1 B_1$
  - $C_1 = L_1 + L_2 - 2L_1 L_2$
  - $L_4 = L_1 L_2$
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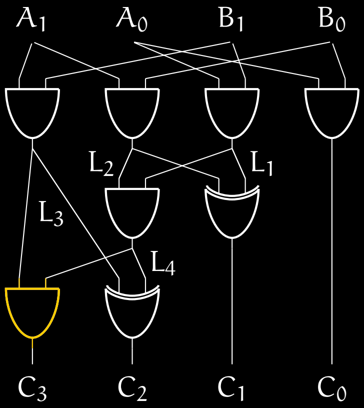

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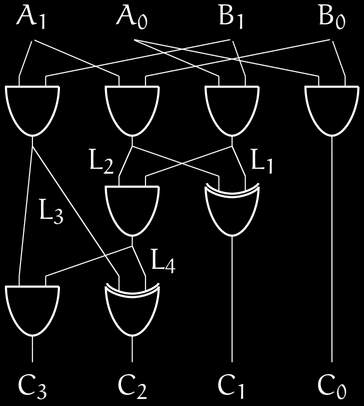

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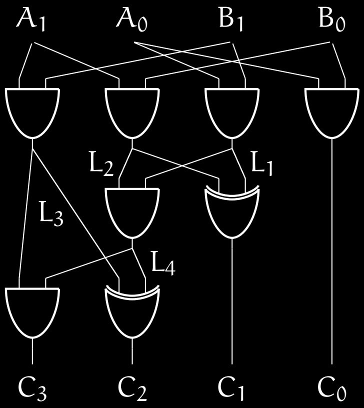

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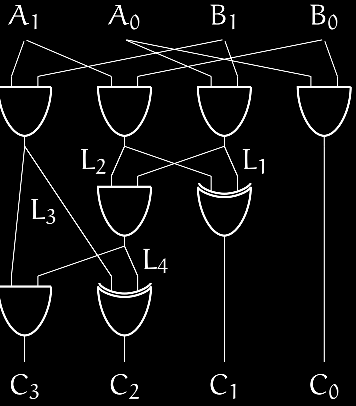

  - $C_0 - A_0 B_0 = 0$
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  - $L_2 - A_1 B_0 = 0$
  - $L_3 - A_1 B_1 = 0$
  - $C_1 - L_1 - L_2 + 2L_1 L_2 = 0$
  - $L_4 - L_1 L_2 = 0$
  - $C_2 - L_3 - L_4 + 2L_3 L_4 = 0$
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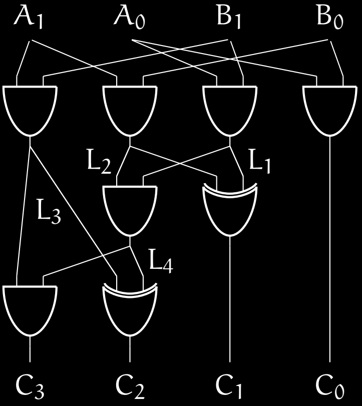

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  - $L_4 - L_1 L_2$
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- Correctness thus reduces to ideal membership test.

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- These cofactors $q_1, \ldots, q_m$ can serve as certificate of the ideal membership.
- This is well-known in theory, but not so easy in practice: the cofactors can be quite large.

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- We construct a formal proof by tracing the reduction process

```
⋮
* : -b+1-a,       a,            -a*b+a-a^2;
+ : -a*b+a-a^2,   a^2-a,        -a*b;
+ : -a*b,         -c+a*b,       -c;
* : -c,           -1,           c;
⋮
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