How to take a square root of a modular tensor category?

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a joint work with Hao Zheng [arXiv:1704.01447, 1705.01087]

- A well known problem in mathematics:

> *For a given modular tensor category (MTC) $\mathcal{C}$, find a mathematical structure ? such that its "Drinfeld center" gives $\mathcal{C}$, i.e. $\mathcal{C} \simeq Z(?)$.*

It is crucial to the problem of extending a 2+1D Reshetikhin-Turaev TQFT (defined by $\mathcal{C}$) down to points (i.e. a 0-1-2-3 TQFT).

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- For a modular tensor category $\mathcal{M}$, we have $\mathcal{C} \simeq \mathcal{M} \boxtimes \overline{\mathcal{M}} \simeq Z(\mathcal{M})$. $\mathcal{C} \simeq Z(?) \Rightarrow \sqrt{\mathcal{C}} =?$. It certainly reminds us "$\sqrt{-1}$", "$\sqrt{\Delta}$", etc.

Partial results:

- When $\mathcal{C}$ is non-chiral, i.e. $\mathcal{C} = Z(\mathcal{M})$ for a spherical fusion category $\mathcal{M}$, the TQFT can be extended to points, to each of which we assign $\mathcal{M}$. Such a TQFT is called a Turaev-Viro TQFT.

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For a generic MTC, there is no clue from the mathematical side.

Clues from physics for $\mathcal{C} = Z(?)$:

1. Physical meaning of a unitary modular tensor category $\mathcal{C}$:

   In physics, a 2+1D topological order, which is a quantum phase at zero temperature of a gapped manybody systems (such as Fractional Quantum Hall Systems), is described by a pair $(\mathcal{C}, c)$, where

   - $\mathcal{C}$ is a unitary modular tensor category (UMTC) $\mathcal{C}$,
     1. objects in $\mathcal{C}$ are topological excitations (also called anyons);
     2. morphisms are observables (i.e. instantons) on 0+1D world line supported on the excitation.
   - c is a real number called chiral central charge.

2. What is the physical meaning of Drinfeld center?

   Let us first look at the case, in which $\mathcal{C}$ is non-chiral. In this case, the topological order $(\mathcal{C}, 0)$ has gapped boundaries.

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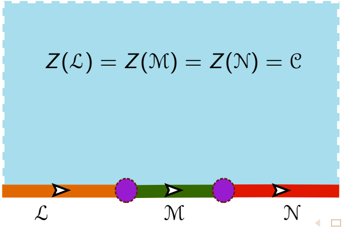
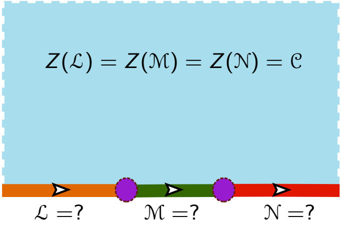
- Boundary-bulk relation:
  1. $Z(\mathcal{M}) = \mathcal{C}$; [Kitaev-K.:11, K.:13]
  2. Two boundaries $\mathcal{M}$ and $\mathcal{N}$ share the same bulk as their Drinfeld center iff they are Morita equivalent [Müger:01,Etingof-Nikshych-Ostrik:08].



When $\mathcal{C}$ is chiral, the topological order $(\mathcal{C}, c)$ has a chiral gapless boundary which is topologically protected.

In 2015, without knowing what a gapless boundary is, K.-Wen-Zheng provide a physical proof of the boundary-bulk relation, i.e. bulk = the center of the boundary:



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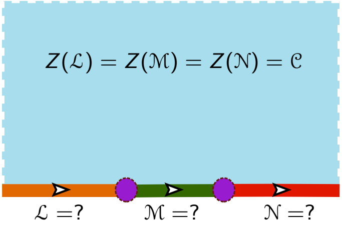

Our mother nature provides a solution to the equation $\mathcal{C} = Z(?)$ when $\mathcal{C}$ is chiral! Only thing remains to do is to read her book.

More precisely, what we need to do is to

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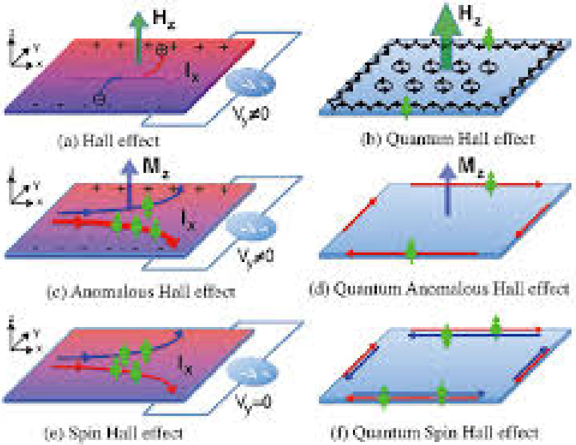
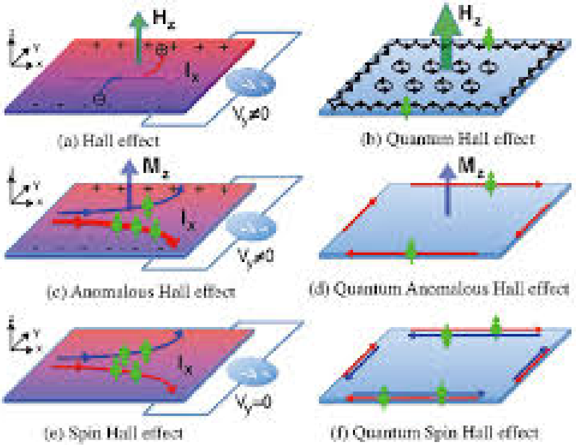
Had physicists already done that?

It was known to physicists that what appear on a gapless boundary is a "chiral conformal field theory" without being very precise [Witten:89, Wen:90's, ...]. The goal of this talk is to make this statement precise.

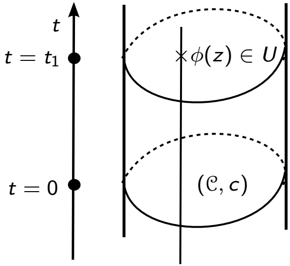


H_z
I_x
$V_y \neq 0$
(a) Hall effect
H_z
(b) Quantum Hall effect
M_z
I_x
$V_y \neq 0$
(c) Anomalous Hall effect
M_z
(d) Quantum Anomalous Hall effect
I_x
$V_y = 0$
(e) Spin Hall effect
(f) Quantum Spin Hall effect



Time axis is missing. “Time is also a phase of matter.”

Observables on the 1+1D world sheet of a gapless boundary of a topological order $(\mathcal{C}, c)$:



1. Monodromy-free chiral fields $\phi(z)$ lives on the 1+1D world sheet.
2. All such chiral fields has OPE, thus form an algebraic structure called "chiral algebra = vertex operator algebra (VOA)", denoted by U = the infinity dimensional space of all chiral fields [Witten:89,Wen:90].

Basic ingredients of a chiral algebra = VOA:

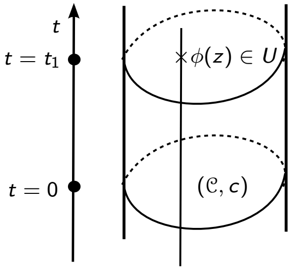
1. U: the space of monodromy-free chiral fields:
$$\phi(z) = \sum_{n\in\mathbb{Z}} \phi_n z^{-n-1};$$

2. $$\phi(z_1)\psi(z_2) \sim \sum_{k<N_{\psi,\phi}} \frac{(\psi_k\phi)(z_2)}{(z_1-z_2)^{k+1}} + \cdots, \qquad k\in\mathbb{Z}.$$

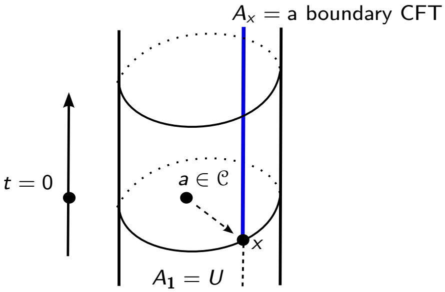
3. $$\phi(z_1)\psi(z_2) \sim \psi(z_2)\phi(z_1);$$

4. a sub-VOA $\langle T\rangle \subset U$, $T(z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$.
$$[L(m), L(n)] = (m-n)L(m+n) + \frac{c}{12}(m^3-m)\delta_{m+n,0}.$$

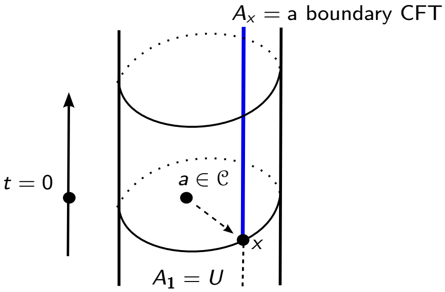
Observables on the 1+1D world sheet of a gapless boundary of $(\mathcal{C}, c)$ includes



1. a VOA;
2. Are there any more observables ?



1. When a bulk excitation $a \in \mathcal{C}$ is moved to the boundary, it creates a "boundary excitation" x, (or a chiral vertex operator by [Wen:90's])



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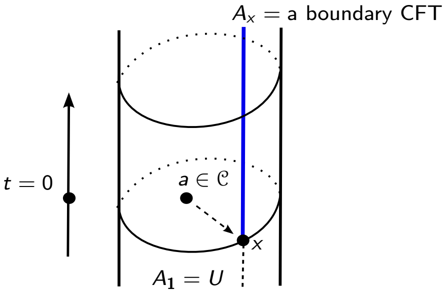
2. The chiral fields living on the world line supported on x are (potentially) different from those in U. We denote the space of all these chiral fields by A_x. These (potentially with monodromy) chiral fields has OPE, which forms an open-string VOA [K.-Huang:03], together with some additional structures, they form a boundary CFT [Cardy:80's].



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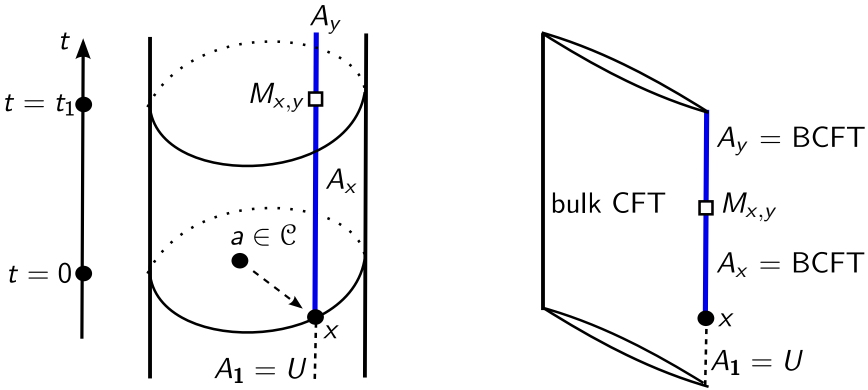
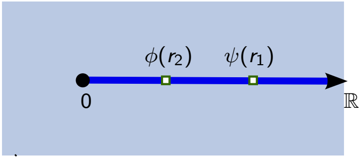
3. $A_{\mathbf{1}} = U$, where $\mathbf{1}$ is the trivial boundary condition.



Anomaly-free principle: A 1+1D boundary-bulk conformal field theory realized by a 1d lattice Hamiltonian model with boundaries should satisfy the mathematical axioms of a boundary-bulk (or open-closed) CFT of all genera, including modular invariance, Cardy condition, etc.

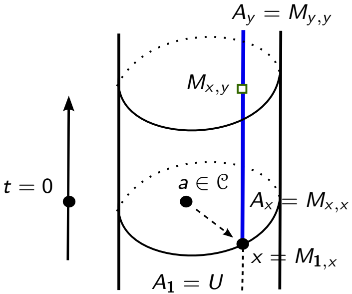
Boundary fields OPE was rigorously defined as an open-string VOA (a non-commutative generalization of a VOA) [Huang-K.:03] :

1. $\phi(r_1) = \sum_{n\in\mathbb{Q}} \phi_n r_1^{-n-1} \in A_x$,

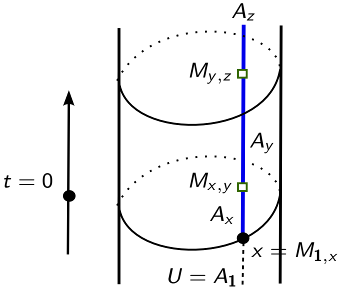


2. $$\psi(r_1)\phi(r_2) \sim \frac{(\psi_k\phi)(r_2)}{(r_1-r_2)^{k+1}} + \cdots, \qquad k \in \mathbb{R}.$$

3. no commutativity: $\psi(r_1)\phi(r_2) \nsim \phi(r_2)\psi(r_1)$.
4. a subalgebra $\langle T\rangle \subset A_x$, where $T(z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$.



1. It is possible that the boundary condition is changed from x to y on the world line at $t = t_1 > 0$.
2. We use $M_{x,y}$ to denote the space of defect fields (boundary condition changing operators, chiral vertex operators) between two boundary CFT's.
3. We have $x = M_{\mathbf{1},x}$ and $A_x = M_{x,x}$.



1. defects fields can be fused (OPE): $M_{y,z} \otimes_{\mathbb{C}} M_{x,y} \to M_{x,z}$.
2. associativity of OPE: $M_{z,w} \otimes_{\mathbb{C}} M_{y,z} \otimes_{\mathbb{C}} M_{x,y} \to M_{x,w}$

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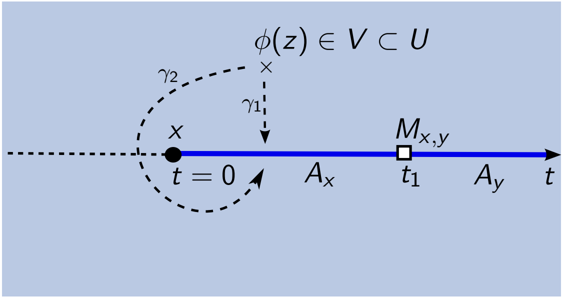
The missing structure:

- identity morphism $\mathrm{id}_x = \iota_{\gamma_i}|_V : \mathbf{1} = V \hookrightarrow A_x = M_{x,x}$.

This is determined by the chiral symmetry of the boundary.

The chiral symmetry V (a VOA) of the boundary:



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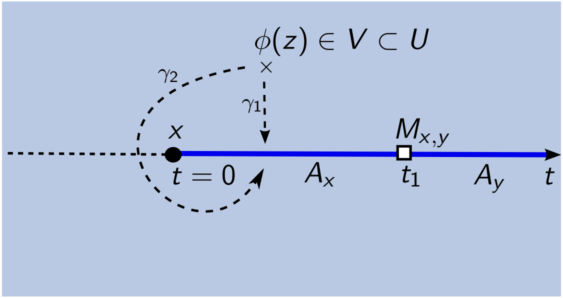

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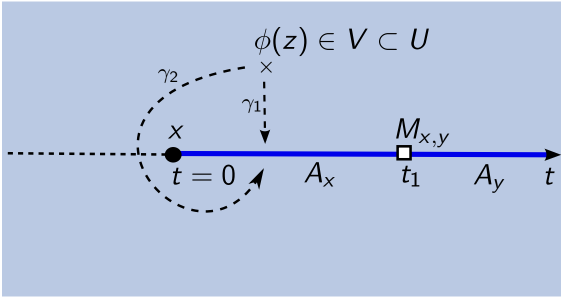

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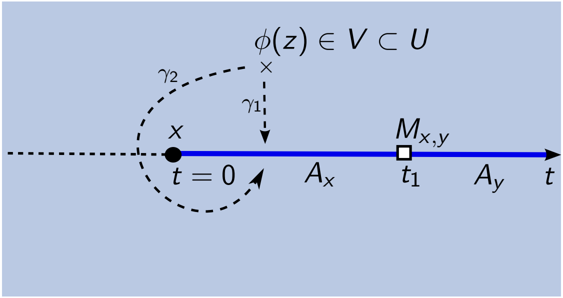

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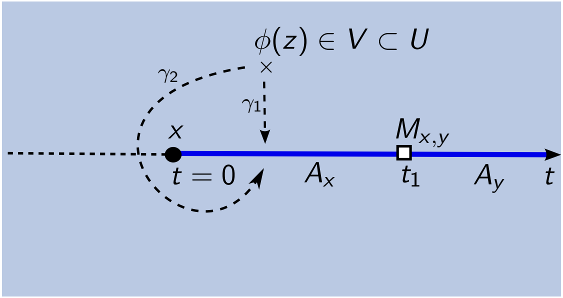
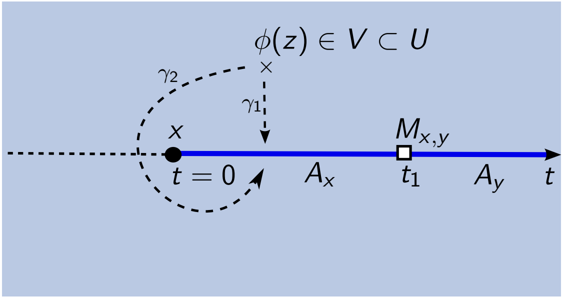
3. V-symmetric condition: $\iota_{\gamma_i}|_V$ are injective and path independent. $\langle T \rangle \subsetneq V \subsetneq U$ in general.



V-symmetric condition:

1. $V \to A_x$ is a path independent injective OSVOA homomorphism;
2. $V \otimes_{\mathbb{C}} M_{x,y} \to M_{x,y}$ is path independent and define a V-module structure on $M_{x,y}$, i.e. $M_{x,y} \in \mathrm{Mod}_V$.



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3. V-action commutes with the fusion $M_{y,z} \otimes_{\mathbb{C}} M_{x,y} \to M_{x,z}$.
$\Rightarrow M_{y,z} \otimes_{\mathbb{C}} M_{x,y} \to M_{x,z}$ is an intertwining operator of V;
(a morphism $M_{y,z} \otimes M_{x,y} \to M_{x,z}$ in Mod_V if Mod_V is monoidal).

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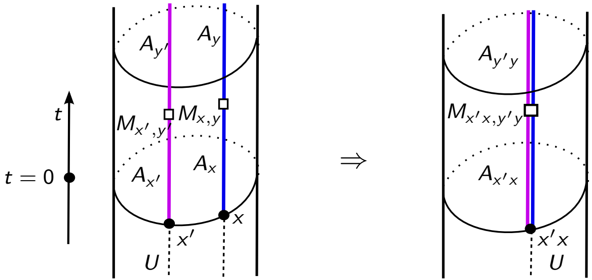
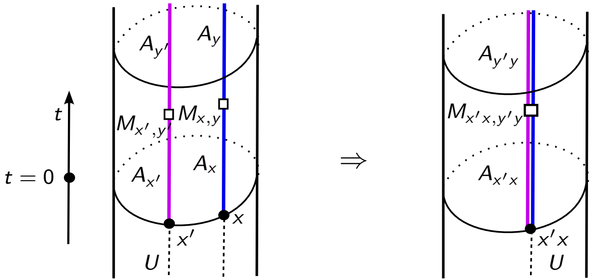
$\mathcal{X}^\sharp$ is a category enriched in Mod_V, or an Mod_V-enriched category.

Assumption: V is a rational VOA. In particular, it means that Mod_V is a modular tensor category (MTC). Huang:04



1. $\otimes : (x', x) \mapsto x'x = x' \otimes x$ and $M_{x',y'} \otimes M_{x,y} \to M_{x'\otimes x, y'\otimes y}$ satisfying some obvious properties.



1. $\otimes : (x', x) \mapsto x'x = x' \otimes x$ and $M_{x',y'} \otimes M_{x,y} \to M_{x' \otimes x, y' \otimes y}$ satisfying some obvious properties.
2. $\otimes$ upgrades $\mathcal{X}^\sharp$ to an Mod_V-enriched monoidal category, a notion which was introduced only recently [Batanin-Markl:12, Morrison-Penneys:17].

Theorem (K.-Zheng, 2017)

All observables on a gapless boundary of a 2+1D topological order $(\mathcal{C}, c)$ can be described by a pair $(V, \mathcal{X}^\sharp)$, where

1. *V is a rational VOA (chiral symmetry);*
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We would like to give an example of such gapless boundaries. Before we do that, we first recall some results of boundary-bulk CFT's.

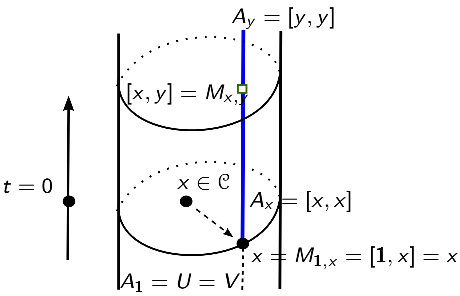
Boundary-bulk (open-closed) CFT with a chiral symmetry V:

Example: $\mathcal{C} = \mathrm{Mod}_V$, $Z(\mathcal{C}) = \mathcal{C} \boxtimes \overline{\mathcal{C}}$ [Müger]

1. $A_{\text{boundary}} = \mathbf{1} = V \in \mathcal{C}$, $A_{\text{bulk}} = Z(\mathbf{1}) = \oplus_i i^* \boxtimes i$.
   [..., Fröhlich-Felder-Fuchs-Schweigert:01]

2. $\forall x \in \mathcal{C}$, $A_{\text{boundary}} = [x, x] = x \otimes x^*$, $Z([x, x]) = Z(\mathbf{1}) = \oplus_i i^* \boxtimes i$.
   [Fuchs-Runkel-Schweigert:04, K.:06, K.-Runkel:07,08]

3. $[x, y] = y \otimes x^*$ defines a V-symmetric wall between boundary CFT's $[x, x]$ and $[y, y]$ [Fröhlich-Fuchs-Runkel-Schweigert:06].



A canonical gapless boundary $(V, \mathcal{C}^\sharp)$ of the topological order $(\mathcal{C}, c)$:

1. "boundary excitations" = bulk excitations = $\mathcal{C}$; $V = U$, $\mathrm{Mod}_V = \mathcal{C}$;
2. $M_{x,y} := [x, y] = y \otimes x^*$ for $x, y \in \mathrm{Mod}_V = \mathcal{C}$;
3. $\mathrm{id}_x : V = \mathbf{1} \to [x, x] = x \otimes x^*$ is given by the duality map;
4. $[y, z] \otimes [x, y] = z \otimes y^* \otimes y \otimes x^* \to z \otimes x^* = [x, z]$.
5. $[x', y'] \otimes [x, y] = y' \otimes x'^* \otimes y \otimes x^* \xrightarrow{1c_{x'^*, y\otimes x^*}1} y' \otimes y \otimes x^* \otimes x'^* = [x' \otimes x, y' \otimes y]$.

Definition (K.-Zheng, 2017)

Let $\mathcal{C}^\sharp$ be a monoidal category enriched over $\mathcal{B}$. A *half-braiding* for an object $x \in \mathcal{C}^\sharp$ is an enriched natural isomorphism $b_x : x \otimes - \to - \otimes x$ between enriched endo-functors of $\mathcal{C}^\sharp$ such that it defines a half-braiding in the underlying monoidal category $\mathcal{C}$. The Drinfeld center of $\mathcal{C}^\sharp$ is a category $Z(\mathcal{C}^\sharp)$ enriched over $\mathcal{B}$ defined as follows:

- an object is a pair (x, b_x), where $x \in \mathcal{C}^\sharp$ and b_x is a half-braiding for x;
- $\hom_{Z(\mathcal{C}^\sharp)}((x, b_x), (y, b_y))$ is the intersection of the equalizers of the diagrams $\hom_{\mathcal{C}^\sharp}(x, y) \rightrightarrows \hom_{\mathcal{C}^\sharp}(x \otimes z, z \otimes y)$ depicted below for all $z \in \mathcal{C}^\sharp$

$$\begin{tikzcd}
\hom_{\mathcal{C}^\sharp}(x, y) \arrow[r, "\otimes\circ(\mathrm{id}_z\otimes\mathrm{Id})"] \arrow[d, "\otimes\circ(\mathrm{Id}\otimes\mathrm{id}_z)"'] & \hom_{\mathcal{C}^\sharp}(z \otimes x, z \otimes y) \arrow[d, "-\circ b_{x,z}"] \\
\hom_{\mathcal{C}^\sharp}(x \otimes z, y \otimes z) \arrow[r, "b_{y,z}\circ -"] & \hom_{\mathcal{C}^\sharp}(x \otimes z, z \otimes y);
\end{tikzcd}$$

- the composition law $\circ$ is induced from that of $\mathcal{C}^\sharp$.

Theorem (K.-Zheng, 2017)

$Z(\mathcal{C}^\sharp) = \mathcal{C}.$

Boundary-bulk relation holds for the canonical gapless boundary!

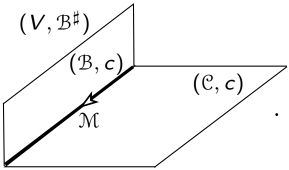
Morrison-Penneys's canonical construction: [2017]

Let $\mathcal{B}$ be a braided monoidal category and $\mathcal{M}$ a monoidal category. Let $f: \overline{\mathcal{B}} \to Z(\mathcal{M})$ be a braided oplax-monoidal functor. Then we have a functor $\odot : \overline{\mathcal{B}} \times \mathcal{M} \to Z(\mathcal{M}) \times \mathcal{M} \to \mathcal{M}$. There is a canonical construction of a $\mathcal{B}$-enriched monoidal category $\mathcal{M}^\sharp$ from the pair $(\mathcal{B}, \mathcal{M})$:

- objects in $\mathcal{M}^\sharp$ are objects in $\mathcal{M}$, i.e. $Ob(\mathcal{M}^\sharp) := Ob(\mathcal{M})$;
- For $x, y \in \mathcal{M}$, $\hom_{\mathcal{M}^\sharp}(x, y) := [x, y]$ in $\overline{\mathcal{B}}$ (or in $\mathcal{B}$);
- $\mathrm{id}_x : \mathbf{1}_{\mathcal{B}} \to [x, x]$ is the morphism in $\mathcal{B}$ canonically induced from the unital action $\mathbf{1}_{\mathcal{B}} \odot x \simeq x$;
- $\circ : [y, z] \otimes [x, y] \to [x, z]$ is the morphism canonically induced from the action $([y, z] \otimes [x, y]) \odot x \to [y, z] \odot y \to z$.
- $\otimes : [x', y'] \otimes [x, y] \to [x' \otimes x, y' \otimes y]$ is the morphism in $\mathcal{B}$ canonically induced from the action

$$\begin{aligned}
([x', y'] \otimes [x, y]) \odot x' \otimes x &= \phi_{\mathcal{M}}([x', y'] \otimes [x, y]) \otimes x' \otimes x \\
&\to \phi_{\mathcal{M}}([x', y']) \otimes \phi_{\mathcal{M}}([x, y]) \otimes x' \otimes x \\
&\xrightarrow{\mathrm{Id} \otimes b_{\phi_{\mathcal{M}}([x,y]), x'} \otimes \mathrm{Id}_x} \phi_{\mathcal{M}}([x', y']) \otimes x' \otimes \phi_{\mathcal{M}}([x, y]) \otimes x \to y' \otimes y.
\end{aligned}$$

More general gapless boundaries of the 2+1D bulk phase $(\mathcal{C}, c)$:

$(V, \mathcal{B}^\sharp) \boxtimes_{(\mathcal{B},c)} \mathcal{M} \quad \to$  .

$$(V, \mathcal{B}^\sharp) \boxtimes_{(\mathcal{B},c)} \mathcal{M} = (V, \mathcal{B}, \mathcal{M}) = (V, \mathcal{M}^\sharp)$$

where $\mathcal{M}^\sharp$ is the enriched monoidal category determined the pair $(\mathcal{B}, \mathcal{M})$ via the canonical construction. [Morrison-Penneys:17]

Corollary (K.-Zheng, 2017)

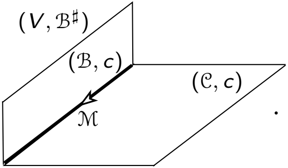
Boundary-bulk relation: $Z(\mathcal{M}^\sharp) = Z(\mathcal{B}, \mathcal{M}) = \mathcal{C}$.

Remark: The mathematical description of a gapped boundary, i.e. a unitary fusion category $\mathcal{M}$, is automatically included in that of a gapless edge, i.e. $(V, \mathcal{B}, \mathcal{M})$, as a special case with $V = \mathbb{C}$ and $\mathcal{B} = \mathbf{H}$, where $\mathbb{C}$ is viewed as the trivial VOA with zero central charge.

$$\mathcal{M} = (\mathbb{C}, \mathbf{H}, \mathcal{M}).$$

Therefore, we have obtained a unified mathematical theory of both gapped and gapless boundaries of 2+1D topological orders.

We argue that all gapless/gapped boundarys are obtained in this way.



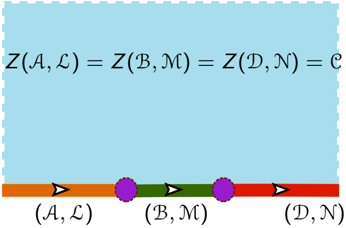
Boundaries of $(\mathcal{C}, c)$ are classified by triples $(V, \mathcal{B}, \mathcal{M})$, where

1. V is rational VOA such that $\mathcal{B} = \mathrm{Mod}_V$ is a UMTC;
2. $\mathcal{M}$ is a unitary fusion category equipped with a braided equivalence $\phi_{\mathcal{M}} : \overline{\mathcal{B}} \boxtimes \mathcal{C} \xrightarrow{\simeq} Z(\mathcal{M})$.

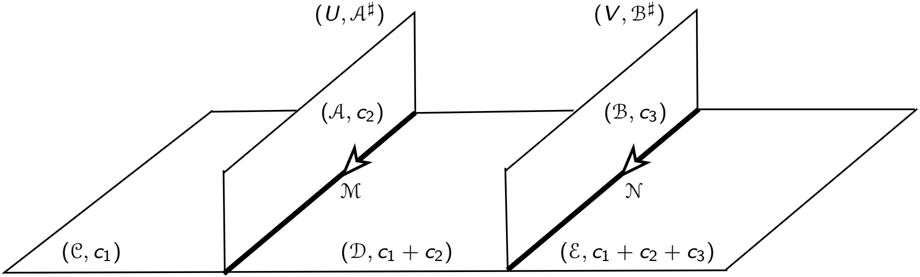
or equivalently, by a pair (V, A), where A is a Lagrangian algebra in $\overline{\mathrm{Mod}_V} \boxtimes \mathcal{C}$.

A gapless/gapped boundary of a given bulk $(\mathcal{C}, c)$ is described by a triple $(V, \mathcal{B}, \mathcal{M})$. We have boundary-bulk relation:

1. $Z(\mathcal{B}, \mathcal{M}) = \mathcal{C}$; [K.-Zheng, 2017]
2. Two boundaries $(\mathcal{A}, \mathcal{L})$ and $(\mathcal{B}, \mathcal{M})$ share the same bulk as their Drinfeld center iff they are Morita equivalent [Zheng, 2017].



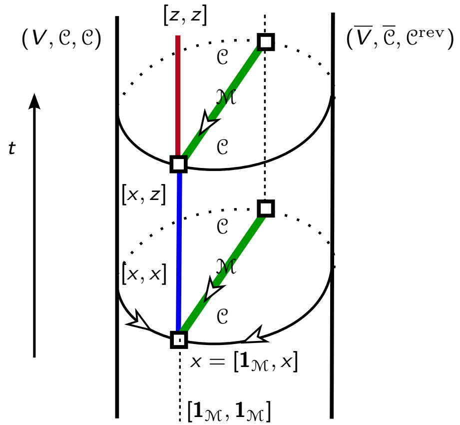
This automatically include a classification of gapped/gapless domain walls between two bulk phases.



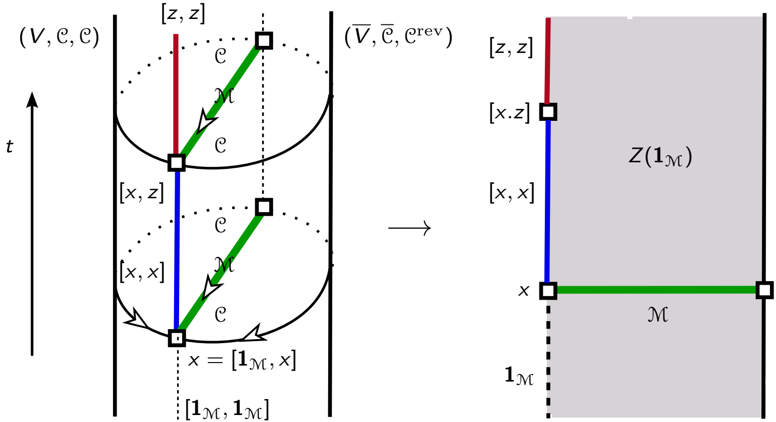
$$(U, \mathcal{A}, \mathcal{M}) \boxtimes_{(\mathcal{D}, c_1+c_2)} (V, \mathcal{B}, \mathcal{N}) = (U \otimes_{\mathbb{C}} V, \mathcal{A} \boxtimes \mathcal{B}, \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}),$$

This is a powerful formula (factorization homology) allows us to compute and construct non-chiral gapless boundaries/walls, which have even wider applications.

As an example, we will show how to recover modular invariant bulk CFT's from 2+1D topological orders.



Modular invariant bulk CFT's from 2+1D topological orders:



$$(V, \mathcal{C}, \mathcal{C}) \boxtimes_{(\mathcal{C},c)} (\mathbb{C}, \mathbf{H}, \mathcal{M}) \boxtimes_{(\mathcal{C},c)} (\overline{V}, \overline{\mathcal{C}}, \mathcal{C}^{\mathrm{rev}}) = (V \otimes_{\mathbb{C}} \overline{V}, \mathcal{C} \boxtimes \overline{\mathcal{C}}, \mathcal{M}).$$

$$(V, \mathcal{C}, \mathcal{C}) \boxtimes_{(\mathcal{C},c)} (\mathbb{C}, \mathbf{H}, \mathcal{M}) \boxtimes_{(\mathcal{C},c)} (\overline{V}, \overline{\mathcal{C}}, \mathcal{C}^{\mathrm{rev}}) = (V \otimes_{\mathbb{C}} \overline{V}, \mathcal{C} \boxtimes \overline{\mathcal{C}}, \mathcal{M}).$$

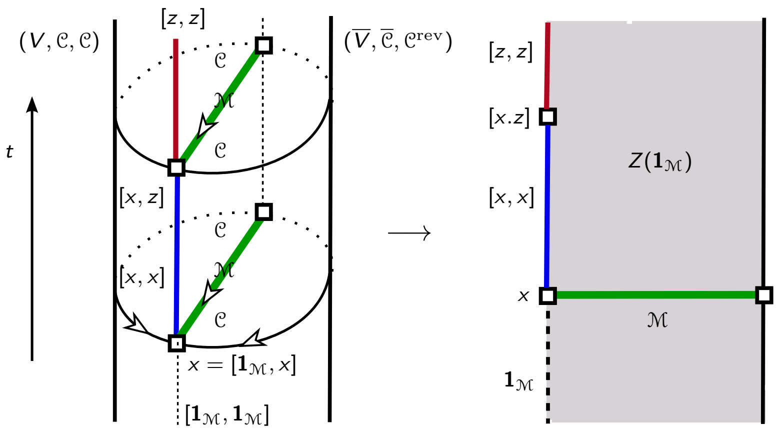
$(V \otimes_{\mathbb{C}} \overline{V}, \mathcal{C} \boxtimes \overline{\mathcal{C}}, \mathcal{M})$ is a gapless domain wall between two trivial phases:

1. $U = [\mathbf{1}_{\mathcal{M}}, \mathbf{1}_{\mathcal{M}}] \in \mathcal{C} \boxtimes \overline{\mathcal{C}}$;
2. If $\mathcal{M} = \mathcal{C}$, $[\mathbf{1}_{\mathcal{M}}, \mathbf{1}_{\mathcal{M}}] = \oplus_i\, i \boxtimes i^*$ is a Lagrangian algebra in $\mathcal{C} \boxtimes \overline{\mathcal{C}}$, which is nothing but the famous charge conjugate modular invariant bulk CFT.
3. $\mathcal{M} \neq \mathcal{C}$, $[\mathbf{1}_{\mathcal{M}}, \mathbf{1}_{\mathcal{M}}]$ is a different Lagrangian algebra in $\mathcal{C} \boxtimes \overline{\mathcal{C}}$ or a different modular invariant bulk CFT.

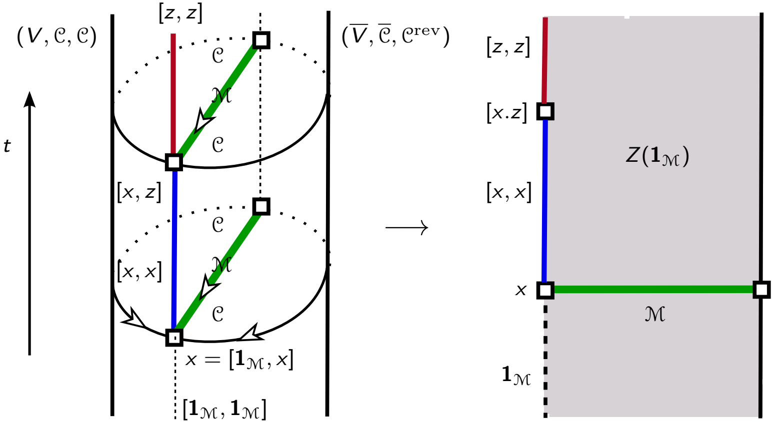
This gapless wall $(V \otimes_{\mathbb{C}} \overline{V}, \mathcal{C} \boxtimes \overline{\mathcal{C}}, \mathcal{M})$ between two trivial phases provides a physical explanation of the one-to-one correspondences among the following three sets:

1. the set of gapped walls between $(\mathcal{C}, c)$ and $(\mathcal{C}, c)$,
2. the set of Lagrangian algebras in $Z(\mathcal{C})$,
3. the set of modular-invariant bulk CFT's in $\mathcal{C} \boxtimes \overline{\mathcal{C}}$.

$$\mathcal{M} \mapsto [\mathbf{1}_{\mathcal{M}}, \mathbf{1}_{\mathcal{M}}] \in \mathcal{C} \boxtimes \overline{\mathcal{C}}.$$



The observables on the 0+1D world line of the boundary of the domain wall $\mathcal{M}$ are described by an enriched category $\mathcal{M}^\sharp = (\mathcal{M}, \mathcal{M})$. $\hom_{\mathcal{M}^\sharp}(x, y) := [x, y] = y \otimes x^* \in \mathcal{M}$.



$$(V, \mathcal{C}, \mathcal{C}) \boxtimes_{(\mathcal{C},c)} (\mathbb{C}, \mathbf{H}, \mathcal{M}) \boxtimes_{(\mathcal{C},c)} (\overline{V}, \overline{\mathcal{C}}, \mathcal{C}^{\mathrm{rev}}) = (V \otimes_{\mathbb{C}} \overline{V}, \mathcal{C} \boxtimes \overline{\mathcal{C}}, \mathcal{M}).$$

This simple formula recovers and encodes all boundary-bulk CFT's over V, including all its ingredients.

Conclusions:

1. We have found a solution to the equation $Z(?) = \mathcal{C}$.
2. We have found a unified mathematical theory of gapless/gapped boundaries of all 2+1D topological order. It leads to a classification theory of all gapless/gapped boundaries and defects of codimension 1 and 2, for 2+1D topological orders.
3. We have shown that this theory can also be used to study non-chiral gapless boundaries/walls.

Outlooks:

1. It opens the way to study gapless boundaries for symmetry enriched topological orders and symmetry enriched topological orders.
2. It gives us a clue how to construct lattice models to realize all chiral 2+1D topological orders (generalizing Levin-Wen models).
3. It provides a systematic way to study topological phase transitions.

Thank you !