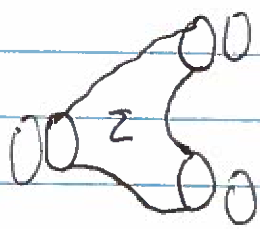
Poisson sigma models and the symplectic category

<u>Plan</u>: Describe from the BV-BFV perspective, how PSM with boundary produces groupoid objects in the "<u>symplectic category</u>".

Joint w. A. Cattaneo, A. Cattaneo - A. Weinstein.

<u>Classical BV-BFV</u> (dim 2)

$\mathcal{F}_{BV\text{-}BFV}: 2\text{-}Cob \longrightarrow Sympl.$



$\longmapsto \mathcal{F}_{\partial,\omega} \otimes \mathcal{F}_{\partial\omega} \otimes \mathcal{F}_{\partial\bar{\omega}}.$

$\partial\Sigma_{in} \quad \partial\Sigma_{in} \quad \partial\Sigma_{out}$

$\Sigma \longmapsto L_\Sigma \subseteq \mathcal{F}_{\partial\omega} \oplus \mathcal{F}_{\partial\omega} \oplus \overline{\mathcal{F}_{\partial\omega}}$

$\quad$ ↳ Evolution relation.

$S \in \mathcal{F}un(\Sigma) \quad \{\varphi : \delta S = 0\} = \underline{EL(\Sigma)}$

$P: \Sigma \longrightarrow \partial\Sigma$

$P_* EL(\Sigma) = L_\Sigma$.

<u>Obs</u>: L_Σ is isotropic.

<u>Q</u>: is L_Σ Lagrangian?

<u>Ex</u>: • Classical mechanics Yes.

• Wave equation: No.

• Poisson sigma model: Yes (Cattaneo-C)

$Obj(Sym) \ni (V, \omega)$. Banach weak symplectic spaces

$Mor(Sym)$ Lagrangian correspondences

General theory

- $C_{\partial\Sigma}$ is coisotropic in $\mathcal{F}_\partial$
- L_Σ is isotropic

Q: Can we guarantee that L_Σ is Lagrangian?
Not always

Ex (Yes). $d=1$ Newtonian mechanics on N
$[a,b]$ $L(q,v) = \frac{1}{2} m\|v\|^2 - V(q)$

$\Phi_L : TN \longrightarrow T^*N$.

$C_\partial = TN$

$\omega_\partial = \Phi^*\omega_{can} = d\alpha$, $\boxed{\alpha = \frac{\partial L}{\partial v^i} dq^i}$

$$L_{[t_a,t_b]} = (\Phi_L^{-1} \times \Phi_L^{-1})\left(\mathrm{graph}\left(\Phi_{H_L}\big|_{t_a}^{t_b}\right)\right)$$

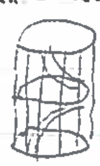
L_∂ are graphs of symplectomorphisms & they compose well

$L_{[t_0,t_2]} = L_{[t_1,t_2]} \circ L_{[t_0,t_1]}$.

Ex No. Wave relations (Cattaneo-Mnev).

Free boson: $S(\phi) = \frac{1}{2}\int_M d\phi \wedge * d\phi$ $g = dx\,dy - y\,dx^2$

$M = S^1 \times [0,1]$ with Misner metric



$L_{\partial M}$ is isotropic but not Lagrangian

Ex: Poisson sigma model. (Yes) (Cattaneo. C)
- (Ikeda, Strobl,)

AKSZ: $(M,\pi) \longmapsto T^*[1]M$.
$(\Sigma, \partial\Sigma) \longrightarrow T^*[1]\Sigma$. $\mathcal{O}(T^*[1]\Sigma) \simeq \Omega^\bullet(\Sigma)$

$\mathcal{F}_\Sigma = \mathrm{Map}(T^*[1]\Sigma, T^*[1]M)$ $\mathcal{O}(T^*[1]M) \simeq \mathrm{Poly}^\bullet(M)$

From AKSZ transgression. & $gh=0$

$$S_{PSM}(X,\eta) = \int_\Sigma \eta\wedge dX + \frac{1}{2}\pi^{ij}(X)\,\eta_i\wedge\eta_j. \text{ dual}_\Sigma \qquad X\in Map(\Sigma, M),\quad \eta\in\Gamma(Hom(T\Sigma, X^*(T^*M)))$$

$$S^{\partial}_{PSM}(X,\eta) = \int_I \eta\wedge dX + \frac{1}{2}\pi^{ij}(X)\,\eta_i\wedge\eta_j$$

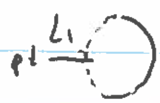
$$\omega_{\partial\Sigma} = \int_I \delta\eta\wedge\delta X \in \omega_{can} \text{ in } T^*Map(I,M)$$

$C_\partial \subseteq F_{\partial M} := \{$ Lie algebroid-paths $\}$
T^*M-paths

$$\begin{array}{ccc} (T[0,1] & \xrightarrow{(d)} & (T^*M, \pi^\#) \\ \downarrow & & \downarrow \\ [0,1] & \longrightarrow & M \end{array}$$

Lie algebroid morphism.

<u>Theorem</u> (~~Co~~) L_1: pt $\longrightarrow$



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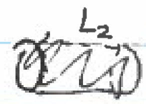
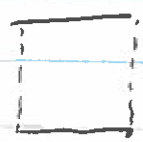
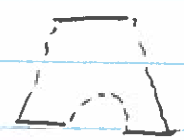
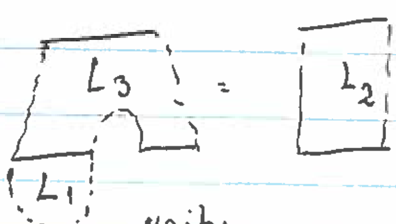


L_2:

L_3


are Lagrangian, & they obey groupoid-like axioms



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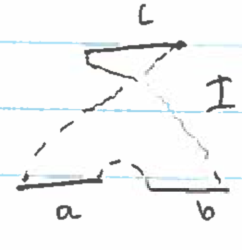
unity

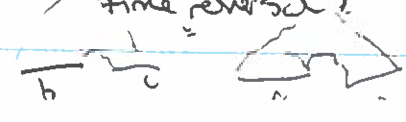
assoc

<u>cyclicity</u>



$I\circ L_3$ ($\bar{I}$ antisymplectomorph / time reversal)

=



$m(a,b,c)=1$
$\Rightarrow m(b,c,a)=1$
$\Rightarrow m(c,a,b)=1$

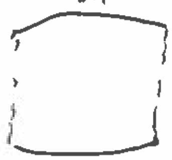
Def. A relational symplectic groupoid is the data of

1) A weak symplectic manifold $\mathcal{G}$
2) A immersed Lagrangian submanifold of $\mathcal{G}\times\mathcal{G}\times\mathcal{G}$
3) An anti-symplectomorphism I

satisfying groupoid-like axioms.

Obs. L_2 :

γ_1



γ_2

is T^*M-homotopy between γ_1, γ_2

i.e. $T\square \longrightarrow T^*M$

$\gamma_1 = T^{-} \longrightarrow T^*M$

$\gamma_2 \quad T\_ \longrightarrow T^*M$

$T|\_| = T[\,] =$ triv.

Proof 1 Brute force (Thesis).
∞-dim sympl. geometry.

Proof 2 Split Lagrangians.

Def. (Tul): A subspace L of a weak symplectic space is S.L if L is isotropic and it has an isotropic complement.

Obs S.L $\Rightarrow$ Lagrangian (maximally isotropic)
$\cancel{\Leftarrow}$

Counterexample: $V = \ell_2 \oplus_{Z} \ell_2$ twisted sum of ℓ_2-spaces
$Z \subseteq V$ is max. isotropic

By (Kalton-Swanson '82). V has no split Lagrangian subspaces

Ongoing w. Cattaneo-Weinstein: split-symplectic category
composition & reduction

Thm - The relational s. groupoid is a groupoid object in $\mathrm{Symp}^{\mathrm{split}}$.

The road not yet taken

- $\mathrm{Symp} \longmapsto \mathrm{Symp}^{ext}$ extended symplectic category
  relational symplectic groupoid as ∞ groupoid.
  $\hookrightarrow$ PSM $\longrightarrow$ Fully extended PSM.

Conj (Contreras - Scheimbauer)
The relational sympl groupoid produces a CY. object in Symp^{∞}
then PSM is an open-closed ext. TFT.