Some results about entropic transport

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Aim of the talk

Aim

- advertising some results about entropic transport
- while providing a new proof of a standard result

HWI

- relative entropy: $H(p|r) := \int \log(dp/dr)\, dp \in [0, \infty]$

- $m = e^{-V}\, \mathrm{Leb} \in \mathrm{P}(\mathbb{R}^n)$, Hess $V \geq \kappa \mathrm{Id}$, $\kappa \in \mathbb{R}$
- $\mu_0, \mu_1, \nu \in \mathrm{P}(\mathbb{R}^n)$

- transport cost: $W_2^2(\mu_0, \mu_1) = \inf_\pi \int_{\mathbb{R}^n \times \mathbb{R}^n} |y - x|^2\, \pi(dxdy)$

- Fisher information: $I(\nu|m) := \int_{\mathbb{R}^n} |\nabla \log(d\nu/dm)|^2\, d\nu$

HWI* inequality [Otto-Villani]

$$H(\mu_1|m) - H(\mu_0|m) \leq W_2(\mu_0, \mu_1)\sqrt{I(\mu_1|m)} - \kappa W_2^2(\mu_0, \mu_1)/2, \quad \forall \mu_0, \mu_1$$

- HWI: $H(\nu|m) \leq W_2(\nu, m)\sqrt{I(\nu|m)} - \kappa W_2^2(\nu, m)/2, \quad \forall \nu$
- Talagrand: $\kappa W_2^2(\nu, m)/2 \leq H(\nu|m), \quad (\kappa > 0), \quad \forall \nu$
- log-Sobolev: $H(\nu|m) \leq I(\nu|m)/(2\kappa), \quad (\kappa > 0), \quad \forall \nu$

HWI

HWI* inequality

$$H(\mu_1|m) - H(\mu_0|m) \leq W_2(\mu_0, \mu_1)\sqrt{I(\mu_1|m)} - \kappa W_2^2(\mu_0, \mu_1)/2, \quad \forall \mu_0, \mu_1$$

- $h(1) - h(0) = h'(1) - \int_0^1 t h''(t)\, dt$
- $t \mapsto h(t) = H(\mu_t|m), \qquad (\mu_t)_{0 \leq t \leq 1}$: displacement interpolation

remainder of the talk: "stochastic" proof of HWI*

replace displacement interpolations by entropic interpolations

1. Schrödinger's problem: entropic interpolations
2. W_2 : slowing down
3. I: entropic actions
4. convexity of $t \mapsto H(\mu_t|m)$: equations of motion

Entropic interpolations

- state space: $\mathcal{X}$
- reference probability measure: $r \in \mathrm{P}(\mathcal{X})$
- particle system: $(\zeta_1, \ldots, \zeta_N) \sim \text{iid}(r)$
- empirical measure: $\hat{\zeta}^N := N^{-1} \sum_{1 \le i \le N} \delta_{\zeta_i} \in \mathrm{P}(\mathcal{X})$
- observe: $\hat{\zeta}^N(\mathrm{obs}) \in \mathcal{C}$

conditional LLN

"knowing that $\hat{\zeta}^N \in \mathcal{C}$", $\hat{\zeta}^N \underset{N \to \infty}{\longrightarrow} p^{\mathcal{C}}$, a.s.

- who is $p^{\mathcal{C}}$?

Entropic interpolations

conditional LLN

$$\text{“knowing that } \hat{\zeta}^N \in \mathcal{C}\text{”}, \quad \hat{\zeta}^N \underset{N\to\infty}{\longrightarrow} p^{\mathcal{C}}, \quad \text{a.s.}$$

- $p^{\mathcal{C}}$: unique (if $\mathcal{C}$ is convex) solution of

$$H(p|r) \to \min; \qquad p \in \mathcal{C}$$

Sanov's theorem

$$\mathbb{P}(\hat{\zeta}^N \in \mathcal{O}) \underset{N\to\infty}{\asymp} \exp\left(-N \inf_{p\in\mathcal{O}} H(p|r)\right), \qquad \mathcal{O} \subset \mathrm{P}(\mathcal{X})$$

Entropic interpolations

heat bath

- state space: $\mathbb{R}^n$
- $\Omega = \{\mathrm{paths}\}$
- evolution: $R \in \mathrm{P}(\Omega)$, Markov, $L = (-\nabla V \cdot \nabla + \Delta)/2$
- equilibrium: $m = e^{-V}\,\mathrm{Leb}$

- $\mathsf{Z} = (\mathsf{Z}_t)_{t\geq 0}$: $\mathsf{Z}_t = \mathsf{Z}_0 - \frac{1}{2}\int_0^t \nabla V(\mathsf{Z}_s)\,ds + \mathsf{W}_t$,
  W: Brownian motion
- R^{μ_0}: $\mathsf{Z}_0 \sim \mu_0 \in \mathrm{P}(\mathbb{R}^n)$, $\mathsf{Z} \sim R^{\mu_0} \in \mathrm{P}(\Omega)$
- $R^m = R$ is reversible

heat flow

- $R_t^{\mu_0} = \mu_0 e^{tL} \in \mathrm{P}(\mathbb{R}^n)$, $t \geq 0$

Entropic interpolations

- N particles travel in the heat bath
- no interaction
- $N \to \infty$

particles

- $(Z^1, \dots, Z^N) \sim (R^{\mu_0})^{\otimes N}$
- $\widehat{Z}^N := N^{-1} \sum_{1 \le i \le N} \delta_{Z^i}$, random values in $\mathrm{P}(\Omega)$

- law of large numbers: $\lim_{N\to\infty} \widehat{Z}^N = R^{\mu_0}$, a.s.
  $\lim_{N\to\infty} \widehat{Z}^N_t = R^{\mu_0}_t = \mu_0 e^{tL}$, a.s.
- time interval: $[0, 1]$

Entropic interpolations

Schrödinger's question (a thought experiment)

Suppose that you observe $\widehat{Z}_1^N(\mathrm{obs}) \simeq \mu_1$, with μ_1 far from the expected profile $R_1^{\mu_0} = \mu_0 e^L$. What is the most likely behavior of $(\widehat{Z}_t^N)_{0\le t\le 1}$?

Schrödinger's answer

1. solve: $H(P|R^{\mu_0}) \to \min, \quad P \in \mathrm{P}(\Omega): \ P_1 = \mu_1$
2. take: $\mu_t = P_t,\ 0 \le t \le 1.$

- $H(P|R^{\mu_0}) = H(P|R) - H(\mu_0|R_0)$

Schrödinger's problem

$$H(P|R) \to \min, \qquad P \in \mathrm{P}(\Omega): \ P_0 = \mu_0,\ P_1 = \mu_1 \qquad \text{(S)}$$

entropic interpolation (definition)

$$[\mu_0, \mu_1]^R = (\mu_t)_{0\le t\le 1}, \qquad \mu_t := P_t,\ 0 \le t \le 1, \quad P = \mathrm{sol(S)}$$

Slowing down

slowing down

- $0 < \epsilon \leq 1, \qquad \epsilon \to 0$
- $Z_t^\epsilon := Z_{\epsilon t},\ 0 \leq t \leq 1$

- $b := -\nabla V/2$
- R: $\qquad dZ_t = b(Z_t)\ dt + dW_t, \qquad\qquad Z_0 \sim m$
- R^ϵ : $\quad dZ_t^\epsilon = \epsilon b(Z_t^\epsilon)\ dt + \sqrt{\epsilon} dW_t, \quad Z_0^\epsilon \sim m$
- $L^{R^\epsilon} = \epsilon L = \epsilon(b \cdot \nabla + \Delta/2)$
- R^ϵ is m-reversible

- $$\lim_{\epsilon \to 0} R^\epsilon(\cdot \mid X_0 = x, X_1 = y) = \delta_{\gamma^{xy}}$$

Slowing down

ϵ-Schrödinger problem

$$\epsilon H(P|R^\epsilon) \to \min; \quad P \in \mathrm{P}(\Omega): \ P_0 = \mu_0, \ P_1 = \mu_1 \qquad (\mathsf{S}^\epsilon)$$

Monge-Kantorovich problem

$$E_P \int_{[0,1]} |\dot{X}_t|^2/2 \, dt \to \min; \quad P \in \mathrm{P}(\Omega): \ P_0 = \mu_0, \ P_1 = \mu_1 \qquad (\mathsf{MK})$$

- $\mathrm{sol}(\mathsf{S}^\epsilon) =: P^\epsilon, \qquad \mathrm{sol}(\mathsf{MK}) =: P$

[Mikami, L.]

$$\Gamma\text{-}\lim_{\epsilon\to 0}(\mathsf{S}^\epsilon) = (\mathsf{MK})$$

$$\lim_{\epsilon\to 0^+} \inf(\mathsf{S}^\epsilon) = \inf(\mathsf{MK}) = W_2^2(\mu_0, \mu_1)/2$$

- $\lim_{\epsilon\to 0^+} P^\epsilon := P \in \mathrm{sol}(\mathsf{MK}),$ (subsequence)
- $\lim_{\epsilon\to 0^+} [\mu_0, \mu_1]^\epsilon := [\mu_0, \mu_1]^{\mathrm{MK}},$ (subsequence)

Entropic actions

- $P \in \mathrm{P}(\Omega)$: $\overrightarrow{L}^P = \overrightarrow{v}^P \cdot \nabla + \epsilon\Delta/2$, Markov

stochastic velocities (Nelson)

- forward: $\overrightarrow{v}_t^P(z) := \lim_{h\to 0^+} E_P\left(\frac{X_{t+h}-X_t}{h} \mid X_t = z\right)$
- backward: $\overleftarrow{v}_t^P(z) := \lim_{h\to 0^+} E_P\left(\frac{X_{t-h}-X_t}{h} \mid X_t = z\right)$

- time reversal: $\overleftarrow{v}_t^P = \overrightarrow{v}_{1-t}^{P^*}$, P^* : time-reversed of P
- momentum: $\beta^P := \epsilon^{-1}(v^P - v^R)$, $\beta^P = \beta^{P|R}$

results

if $H(P|R^\epsilon) < \infty$, then:

- $H(P|R^\epsilon) = H(P_0|m) + \epsilon E_P \int_{[0,1]} |\overrightarrow{\beta}^P(t, X_t)|^2/2\ dt$
- $H(P|R^\epsilon) = H(P_1|m) + \epsilon E_P \int_{[0,1]} |\overleftarrow{\beta}^P(t, X_t)|^2/2\ dt$

Entropic actions

stochastic velocities (Nelson)

- current: $v^{\mathrm{cu},P} := (\overrightarrow{v}^P - \overleftarrow{v}^P)/2$
- osmotic: $v^{\mathrm{os},P} := (\overrightarrow{v}^P + \overleftarrow{v}^P)/2$

- momentum: $\beta^P := \epsilon^{-1}(v^P - v^R)$

results

- $\partial_t \mu + \nabla \cdot (\mu v^{\mathrm{cu},P}) = 0,$ $\mu_t := dP_t/d\mathrm{Leb}$ (continuity equation)
- $\beta^{\mathrm{os},P} = \nabla \log \sqrt{\rho},$ $\rho_t := dP_t/dm$ (time reversal)

Fisher information

$$I(\mu|m) := \int_{\mathbb{R}^n} |\nabla \log \rho|^2 \, d\mu = 4 \int_{\mathbb{R}^n} |\beta^{\mathrm{os},P}|^2 \, d\mu$$

Entropic actions

- forward: $\overrightarrow{A}(P|R^\epsilon) := H(P|R^{\epsilon,P_0\rightarrow}) = \epsilon E_P \int_{[0,1]} |\overrightarrow{\beta}^P(t,X_t)|^2/2\ dt$
- backward: $\overleftarrow{A}(P|R^\epsilon) := H(P|R^{\epsilon,\leftarrow P_1}) = \epsilon E_P \int_{[0,1]} |\overleftarrow{\beta}^P(t,X_t)|^2/2\ dt$

entropic action

$$A(P|R^\epsilon) := \left[\overrightarrow{A}(P|R^\epsilon) + \overleftarrow{A}(P|R^\epsilon)\right]/2$$

result

(S^ϵ) is equivalent to:

$$\epsilon A(P|R^\epsilon) \to \min; \quad P \in \mathrm{P}(\Omega) : \ P_0 = \mu_0, \ P_1 = \mu_1$$

- $H(P|R) = H(P^*|R^*) = H(P^*|R)$

Entropic actions

- current: $A^{\mathrm{cu}}(P|R^\epsilon) := \epsilon E_P \int_{[0,1]} |\beta^{\mathrm{cu},P}(t,X_t)|^2/2 \; dt$
- osmotic: $A^{\mathrm{os}}(P|R^\epsilon) := \epsilon E_P \int_{[0,1]} |\beta^{\mathrm{os},P}(t,X_t)|^2/2 \; dt$

a relevant decomposition

$$\begin{aligned}\epsilon A(P|R^\epsilon) &= \epsilon A^{\mathrm{cu}}(P|R^\epsilon) + \epsilon A^{\mathrm{os}}(P|R^\epsilon)\\ &= \int_{[0,1]\times\mathbb{R}^n} |v^{\mathrm{cu},P}|^2/2 \; dt d\mu_t + \epsilon^2/8 \; \int_{[0,1]} I(\mu_t|m)\, dt\end{aligned}$$

- parallelogram identity: $(|\overrightarrow{\beta}|^2 + |\overleftarrow{\beta}|^2)/2 = |\beta^{\mathrm{cu}}|^2 + |\beta^{\mathrm{os}}|^2$
- recall: $\lim_{\epsilon\to 0} \epsilon A(P^\epsilon|R_\epsilon) = W_2^2(\mu_0,\mu_1)/2$
- biblio: Chen-Georgiou-Pavon, Gentil-L-Ripani

Equations of motion

- R : reference Markov measure

result

if it exists, the unique solution P of (S) satisfies

- P is Markov
- there exist f_0, g_1 such that: $P = f_0(X_0)g_1(X_1)\, R$

$$\begin{cases} f_t(z) & := & E_R[f_0(X_0)|X_t = z] \\ g_t(z) & := & E_R[g_1(X_1)|X_t = z] \end{cases}$$

Born decomposition

$$\mu_t = f_t g_t \; m, \quad 0 \le t \le 1$$

- f, g solve forward and backward linear "heat equations"
- biblio: Schrödinger (1932), Zambrini (1986), L. (2014)

Equations of motion

- define: $\overleftarrow{L}^P_t := \overrightarrow{L}^{P^*}_{1-t}$
- assume: R is reversible, $\overrightarrow{L}^R = \overleftarrow{L}^R =: L$
- define: $\Gamma(u,v) := L(uv) - uLv - vLu$ (carré du champ)

result

- P is Markov
- $\overleftarrow{L}^P_t = L + \dfrac{\Gamma(f_t,\cdot)}{f_t}, \quad \overrightarrow{L}^P_t = L + \dfrac{\Gamma(g_t,\cdot)}{g_t}, \quad 0 \le t \le 1$

- h-transform

Equations of motion

f, g solve forward and backward heat equations

$$\begin{cases} (-\partial_t + L)f = 0 \\ f_{|t=0} = f_0 \end{cases} \qquad \begin{cases} (\partial_t + L)g = 0 \\ g_{|t=1} = g_1 \end{cases}$$

- define: $\varphi := \log f, \quad \psi := \log g$
- define: $Bu := e^{-u} L e^{u}$

φ, ψ solve forward and backward HJB equations

$$\begin{cases} (-\partial_t + B)\varphi = 0 \\ \varphi_{|t=0} = \varphi_0 \end{cases} \qquad \begin{cases} (\partial_t + B)\psi = 0 \\ \psi_{|t=1} = \psi_1 \end{cases}$$

Equations of motion

- $\overrightarrow{L}_t^P = L_{\psi_t}, \qquad \overleftarrow{L}_t^P = L_{\varphi_t}$

ψ-representation

$$\begin{cases} (\partial_t + B)\psi &= 0; \quad \leftarrow \psi_1 \\ (-\partial_t + L_{\psi_t})\mu &= 0; \quad \mu_0 \rightarrow \end{cases}$$

φ-representation

$$\begin{cases} (-\partial_t + B)\varphi &= 0; \quad \varphi_0 \rightarrow \\ (\partial_t + L_{\varphi_t})\mu &= 0; \quad \leftarrow \mu_1 \end{cases}$$

time reversal

$$\varphi + \psi = \log \rho$$

Equations of motion

- $L = (b \cdot \nabla + \Delta/2), \qquad Bu = Lu + |\nabla u|^2/2$
- $L^\epsilon := \epsilon L$
- $B^\epsilon u := e^{-u} L^\epsilon e^u = \epsilon B u$

$$\begin{cases} (-\partial_t + \epsilon B)\varphi = 0 \\ \varphi_{|t=0} = \varphi_0 \end{cases} \qquad \begin{cases} (\partial_t + \epsilon B)\psi = 0 \\ \psi_{|t=1} = \psi_1 \end{cases}$$

- $\varphi, \psi \approx O(\epsilon^{-1})$

result

$$\overrightarrow{\beta} = \nabla\psi, \qquad \overleftarrow{\beta} = \nabla\varphi$$

- $\overrightarrow{v}^P = \epsilon b + \epsilon \overrightarrow{\beta}, \qquad \overleftarrow{v}^P = \epsilon b + \epsilon \overleftarrow{\beta},$

Equations of motion

- Γ_n : n^{th} iteration of Γ^L, $(\epsilon = 1)$
- $\Gamma_0(\varphi) = \varphi^2$, $\Gamma_1(\varphi) = |\nabla\varphi|^2$, $\Gamma_2(\varphi) = \sum_{i,j}(\partial_i\partial_j\varphi)^2 + \text{Hess}\, V(\nabla\varphi)$
- $\Gamma_n^{L^\epsilon} = \epsilon^n \Gamma_n$

conserved quantities

$$t \mapsto \int_{\mathbb{R}^n} \Gamma_n(f_t, g_t)\, dm \quad \text{is constant,} \qquad n \geq 0$$

- $n = 0$: $\int_{\mathbb{R}^n} \Gamma_0(f_t, g_t)\, dm = \int_{\mathbb{R}^n} f_t g_t\, dm = 1$, (conservation of mass)
- $n = 1$: $$\begin{aligned}\int_{\mathbb{R}^n} \Gamma(f_t, g_t)\, dm &= -\int_{\mathbb{R}^n} \overrightarrow{\beta}_t \cdot \overleftarrow{\beta}_t\, d\mu_t \\ &= -\int_{\mathbb{R}^n} |\beta_t^{\text{cu}}|^2\, d\mu_t + \int_{\mathbb{R}^n} |\beta_t^{\text{os}}|^2\, d\mu_t\end{aligned}$$

$$t \mapsto \int_{\mathbb{R}^n} |v^{\text{cu}}|^2\, d\mu_t^\epsilon - \epsilon^2/4 \int_{\mathbb{R}^n} I(\mu_t|m)\, d\mu_t \quad \text{is constant}$$

Derivatives of the entropy

- recall: $\beta^{\mathrm{os}} = \nabla \log \sqrt{\rho}$, $\partial_t \mu + \nabla \cdot (\mu v^{\mathrm{cu}}) = 0$
- denote: $h(t) := H(\mu_t^\epsilon | m)$

derivatives of the entropy (1), [L]

$$h'(t) = \epsilon/2 \int_{\mathbb{R}^n} \{\Gamma(\psi_t) - \Gamma(\varphi_t)\}\, d\mu_t$$

- $\epsilon(\Gamma(\psi) - \Gamma(\varphi)) = \epsilon(|\overrightarrow{\beta}|^2 - |\overleftarrow{\beta}|^2) = 4\epsilon \beta^{\mathrm{os}} \cdot \beta^{\mathrm{cu}} = 4\beta^{\mathrm{os}} \cdot v^{\mathrm{cu}}$
- $h'(t) = 2 \int_{\mathbb{R}^n} \beta_t^{\mathrm{os}} \cdot v_t^{\mathrm{cu}}\, d\mu_t = \int_{\mathbb{R}^n} \nabla \log \rho_t \cdot v_t^{\mathrm{cu}}\, d\mu_t$

makes Otto's heuristics rigorous

$$h'(t) \text{ “}=\text{” } \left\langle \mathrm{grad}_{\mu_t}^{W_2} H(\cdot|m), \dot{\mu}_t \right\rangle_{\mu_t}$$

$$|h'(t)| \le \sqrt{I(\mu_t|m)}\, \left[\int_{\mathbb{R}^n} |v_t^{\mathrm{cu}}|^2\, d\mu_t\right]^{1/2}$$

Derivatives of the entropy

derivatives of the entropy (2), [L.]

$$h''(t) = \epsilon^2/2 \ \int_{\mathbb{R}^n} \{\Gamma_2(\psi_t) + \Gamma_2(\varphi_t)\} \, d\mu_t$$

- Hess $V \geq \kappa \mathrm{Id}, \quad \kappa \in \mathbb{R}$
- $\Gamma_2(\varphi) \geq$ Hess $V(\nabla\varphi) \geq \kappa|\nabla\varphi|^2$
- $\epsilon^2/2 \ (\Gamma_2(\psi) + \Gamma_2(\varphi)) \geq \epsilon^2\kappa/2 \ (|\overrightarrow{\beta}|^2 + |\overleftarrow{\beta}|^2)$
  $= \kappa(|v^{\mathrm{cu}}|^2 + \epsilon^2|\beta^{\mathrm{os}}|^2)$

$$h''(t) \geq \kappa \int_{\mathbb{R}^n} |v_t^{\mathrm{cu}}|^2 \, d\mu_t + \kappa\epsilon^2 I(\mu_t|m)/4$$

HWI

$$H(\mu_1|m) - H(\mu_0|m) \leq W_2(\mu_0,\mu_1)\sqrt{I(\mu_1|m)} - \kappa W_2^2(\mu_0,\mu_1)/2$$

sketch of proof

- $h(1) - h(0) = h'(1) - \int_0^1 t h''(t)\, dt$
- $h_\epsilon(t) = H(\mu_t^\epsilon|m), \qquad \epsilon \to 0$

- $h'_\epsilon(1) \leq \left[\int_{\mathbb{R}^n} |v_1^{\mathrm{cu},\epsilon}|^2\, d\mu_1\right]^{1/2} \sqrt{I(\mu_1|m)}$
- $\int_{[0,1]} t h''_\epsilon(t) \geq \kappa \int_{[0,1]\times\mathbb{R}^n} t|v_t^{\mathrm{cu},\epsilon}|^2\, dt d\mu_t^\epsilon + \kappa\epsilon^2 \int_{[0,1]} tI(\mu_t^\epsilon|m)\, dt/4$

- $t \mapsto \int_{\mathbb{R}^n} |v_t^{\mathrm{cu},\epsilon}|^2\, d\mu_t^\epsilon - \epsilon^2/4 \int_{\mathbb{R}^n} I(\mu_t^\epsilon|m)\, d\mu_t^\epsilon$ is constant
- $\lim_{\epsilon\to 0^+} \int_{[0,1]\times\mathbb{R}^n} |v_t^{\mathrm{cu},\epsilon}|^2\, dt d\mu_t^\epsilon = W_2^2(\mu_0,\mu_1)$

- $\lim_{\epsilon\to 0^+} \epsilon^2 \int_{[0,1]} I(\mu_t^\epsilon|m)\, dt = 0$
- put everything together

□

Main features

instructions for use

1. use the two directions of time to compute the time derivatives
2. express the forward and backward entropic actions with $\overrightarrow{\beta}$ and $\overleftarrow{\beta}$
3. use the parallelogram identity to express them in terms of current and osmotic actions
4. slow down the reference process

- the osmotic action is a small perturbation of the current action
- it allows to follow Otto's heuristics with rigorous calculations

Literature

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Literature

take a visit to some webpages

- Mokaplan: https://team.inria.fr/mokaplan/
- Tryphon: http://georgiou.eng.uci.edu/
- Luca's thesis: https://sites.google.com/site/lucanenna/
- CL: http://leonard.perso.math.cnrs.fr/

gracias por su atención

Erwin Schrödinger

The non-physicist finds it hard to believe that really the ordinary laws of physics, which he regards as the prototype of inviolable precision, should be based on the statistical tendency of matter to go over into disorder.