On hyperfiniteness of boundary actions of hyperbolic groups

Marcin Sabok

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This is joint work (in progress) with Jingyin Huang and Forte Shinko.

Definition (δ-hyperbolic space)

Suppose X is a geodesic metric space, $\delta > 0$ and $x, y, z \in X$. A geodesic triangle whose sides are geodesic segments $[x, y]$, $[y, z]$ and $[z, x]$ is called δ-*slim* if any of the three above geodesic segments is in the δ-neighborhood of the two remaining sides.

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In general, the smaller δ is, the more δ-hyperbolic spaces "look like" trees.

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Examples

There are many examples of hyperbolic groups. The free groups F_n are of course hyperbolic. All fundamental groups $\pi_1(M)$ of compact hyperbolic manifolds M are hyperbolic.

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Given a hyperbolic space X with a distinguished point O we identify two geodesic rays γ_1 and γ_2 (write $\gamma_1 \sim \gamma_2$) if there exists a constant $K > 0$ such that

$$d(\gamma_1(t), \gamma_2(t)) < K$$

for all t. The *boundary of* X, denoted ∂X is the set of all $\sim$-classes of geodesic rays in X.

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Thus defined, ∂X is just a set and it carries a natural compact topology.

Definition (Gromov product)

Given three points x, y, z in a hyperbolic space X we define the Gromov product as follows

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Topology on the boundary

Given $p \in \partial X$ and $r > 0$ we define the neighborhood of p as

$$\{q \in \partial X : \exists \gamma \in q, \exists \gamma' \in p \quad \inf_{s,t \to \infty} (\gamma(s), \gamma'(t))_O \geq r\}$$

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With the above topology, the boundary is a compact topological metrizable space.

Boundary of a hyperbolic group

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Suppose Γ is a hyperbolic group and $p \in \partial\Gamma$. Let $\gamma \in p$ be a geodesic ray. For any $g \in \Gamma$ there exists a unique geodesic ray starting at e which hits the geodesic $\gamma'(t) = g \cdot \gamma(t)$. Denote this geodesic ray by $g\gamma$.

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Boundary action

The above $(g, p) \mapsto [g\gamma]_\sim$ induces an action of Γ by homeomorphism on the boundary $\partial\Gamma$ which is called the *boundary action*.

Definition

The *tail equivalence relation* E_t is the equivalence relation defined at $2^{\mathbb{N}}$ as follows:

$$x \; E_t \; y \quad \text{if} \quad \exists n, m \; \forall k \quad x(n+k) = y(m+k)$$

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Theorem (Dougherty–Jackson–Kechris)

The tail equivalence relation is hyperfinite.

Question

Is the boundary action of every hyperbolic group hyperfinite?

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We can provide positive answer for a large class of hyperbolic groups.

Model triangles

Suppose X is a geodesic metric space. Given three points $x, y, z \in X$ and geodesic segments $[x, y], [y, z], [z, x]$ consider a corresponding triangle x', y', z' on the Euclidean plane with the lengths of $[x', y'], [y', z'], [z', x']$ equal to the corresponding lengths $[x, y], [y, z], [z, x]$.

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For any two points $p, q \in [x, y] \cup [y, z] \cup [z, x]$ there exist unique $p', q' \in\in [x, y] \cup [y, z] \cup [z, x]$ which divide the sides in the same proportion.

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Definition

The space X is *CAT(0)* if the for any $x, y, z, p, q \in X$ as above we have $d(p, q) \leq d_e(p', q')$ where d_e is the Euclidean distance.

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The action of a group on a cube complex is *cocompact* if there are finitely many orbits.

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Hyperbolic groups

For example, such a group is hyperbolic if and only if the complex is hyperbolic.

It turns out that many hyperbolic groups act properly and cocompactly on CAT(0) cube complexes

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Theorem (Bergeron–Wise, Kahn–Markovic)

All fundamental groups of hyperbolic closed 3-manifolds admit proper cocompact actions on CAT(0) cube complexes.

Theorem (Huang–S.–Shinko)

If a hyperbolic group Γ acts properly and cocompactly on a CAT(0) cube complex, then the boundary action of Γ on $\partial\Gamma$ is hyperfinite.

Boundary of a complex

Given a proper and cocompact action of a hyperbolic group Γ on a complex X one can define the boundary of this action in a similar way as the boundary of the group.

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Hyperfiniteness

If a hyperbolic group acts properly and cocompactly on a complex, then the this induces an action on the boundary of the complex, which is hyperfinite if and only if the boundary action of the group is hyperfinite.

Theorem (Huang–S.–Shinko)

If a hyperbolic group Γ acts properly and cocompactly on a CAT(0) cube complex X, then the induced action $\Gamma \curvearrowright \partial X$ is hyperfinite.