Analytic and Geometric Aspects of Spectral Theory

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1 Overview of the Field

The BIRS-CMO workshop "Analytic and Geometric Aspects of Spectral Theory" (22w5149) was held at the Casa Matemática Oaxaca from August 15–19, 2022.

Spectral theory has its origins in physics, specifically the study of vibrating objects. When an object such as a drum vibrates, it does so at a particular set of resonance frequencies, with a corresponding set of resonance modes. Mathematically, these resonance frequencies, or more precisely their squares, are eigenvalues of the Laplace operator, and the resonance modes are the corresponding eigenfunctions. So in order to find the resonance frequencies of a given object, one must solve the partial differential equation known as Laplace's equation. This equation is not usually explicitly solvable unless the object has a great deal of symmetry. Nevertheless, it is often possible to study the qualitative behavior of the spectrum – the collection of eigenvalues – and in particular to study the relationship between the spectrum and the geometry of the object. As such the field sits at the intersection of partial differential equations, analysis, and geometry.

There are still many unanswered questions even about eigenvalues of the Laplacian – for example it is not known whether there exist two smoothly bounded planar domains which are isospectral but not isometric. But as with most areas of mathematics, spectral theory has branched out significantly. One may consider other related operators in mathematical physics, such as the heat operator, the Schrödinger operator, and the Dirac operator. There are discrete analogues of each of these operators and so it is possible to formulate analogous problems on graphs. Singular geometries, such as domains with corners and/or conical singularities, have also been studied.

This workshop was organized around a theme of research in groups. Although there were five plenary talks, the majority of the time was spent with participants working in seven groups, each on a new problem. The specific background of each problem is discussed in the "Reports of the Groups" section below.

2 Workshop Structure and Participants

In advance of the workshop, the organizers reached out to a number of outstanding well-established mathematicians to propose problems. Upon securing seven proposals, participants were asked to rank their top three preferences, and were sorted into groups based on the replies. Other factors in sorting groups included:

- A desire to emphasize new collaborations rather than the continuing of existing collaborative relationships.

- Fostering communication and collaboration between participants in Latin America and those outside it.
- Providing a supportive environment for graduate students and recent Ph.Ds with an eye towards helping them build connections to benefit their subsequent careers.

After the groups were chosen, it was announced that the workshop would be held in a hybrid format due to COVID-19. As such, each of the groups had some participants in person and some online. Specifically, there were fourteen participants in person in Oaxaca and the remaining twenty-four were online. Many of the online participants were located in Europe. This caused logistical challenges, particularly with time zones, so groups were encouraged to set their own schedules for group work if they did not want to meet in the late afternoon in Mexico time.

The composition of the groups is given below. Group leaders – those mathematicians who originally proposed each problem – are indicated in bold face, and the graduate students are indicated in italics.

Group 1: Inverse problems and spectral theory of hyperbolic manifolds related to heat kernel rigidity

Irving Calderón (University of Durham), **Gilles Carron** (University of Nantes), Rafael del Río (UNAM), Asma Hassannezhad (University of Bristol), Sergiu Moroianu (Romanian Academy of Sciences), Xuwen Zhu (Northeastern University).

Group 2: Inverse Steklov problems on polygons

Emily Dryden (Bucknell University), **Carolyn Gordon** (Dartmouth College), *Javier Moreno* (Universidad de los Andes), Julie Rowlett (Chalmers), Carlos Villegas-Blas (UNAM-Cuernavaca).

Group 3: Can Dirac boundary conditions (e.g. MIT) be specified using only observable quantities (e.g. currents...)?

Xenia Fedosova (University of Freiburg), **Nadine Grosse** (University of Freiburg), *Edison Leguizamon* (Universidad de los Andes), Alejandro Uribe (University of Michigan), Hanne van den Bosch (University of Chile), *Angela Vargas* (Universidad Católica de Chile).

Group 4: Heat asymptotics for Steklov-type problems on manifolds and domains with singular boundaries

Cipriana Anghel (Simon Stoilow Institute of Mathematics), Daniel Grieser (University of Oldenburg), *Andrés Patiño* (Universidad de los Andes), *Camilo Pérez* (Universidad de los Andes), **Iosif Polterovich** (Université de Montréal).

Group 5: Convergence of Laplacians under singular perturbations

Sabine Bögli (University of Durham), Olivier Bourget (Universidad Catlica de Chile), Vladimir Lotoreichik (Czech Academy of Sciences), **Olaf Post** (University of Trier), Ricardo Weder (UNAM).

Group 6: Scattering theory for difference equations with operator coefficients

Abdon Choque (Universidad Michoacana), David Sher (DePaul University), **Luis Silva** (UNAM), Boris Vertman (University of Oldenburg), Monika Winklmeier (Universidad de los Andes).

Group 7: Scattering theory for differential forms and low energy expansions: low regularity domains, relations to cohomology and harmonic analysis

Clara Aldana (Universidad del Norte), Nelia Charalambous (University of Cyprus), Dmitry Jakobson (McGill University), Xenia Spilioti (University of Göttingen), **Alexander Strohmaier** (University of Leeds), Amir Vig (University of Michigan).

To briefly comment on the overall demographics of the 38 participants, there were 15 women (39%), 12 participants of Latin American origin (32%), and 13 participants working in Latin America (34%).

3 Plenary Activities

3.1 Lectures

Five participants gave lectures, each of one hour. The presenters were asked to give talks that were broadly accessible to graduate students. Talks were well-attended both in person and online.

- Sabine Bögli: Constructing Schrödinger operators with prescribed eigenvalues.
- Rafael del Río: Rank one singular perturbations of selfadjoint operators.
- Xenia Fedosova: Whittaker Fourier type solutions to differential equations arising from string theory.
- Daniel Grieser: The Calderón projector and Dirichlet-Neumann operator for fibred cusp geometries.
- Alejandro Uribe: The asymptotic distribution of eigenvalues of the Dirichlet-to-Neumann operator on the ball.

3.2 Graduate Student Short Talks

On Thursday afternoon, each of the PhD students at the workshop was given a chance to make a short (10-12 minute) presentation. The goal was to help them advertise their ideas and also gain experience and confidence. In all, five graduate students gave talks:

- Cipriana Anghel: Non-local coefficients in the heat asymptotics for real powers of Laplacians.
- Edison Leguizamon: Summability properties of solutions of second order differential equations with complex potentials.
- Javier Moreno: Perturbations of normal operators.
- Andres Felipe Patiño: Nonproper Dissipative Extensions of Operators with Bounded Imaginary Part.
- Camilo Pérez: On quasi-isospectral Schrdinger operators.

3.3 Discussion on Equity, Diversity, and Inclusion

On Tuesday afternoon, we had a one-hour discussion on equity, diversity, and inclusion in mathematics. To avoid cliches, we chose to focus our discussion on a specific topic: ensuring access to career opportunities that do *not* show up on a resume. Studies and surveys have shown that lack of informal mentoring and networking experience can be obstacles to success of students from underrepresented groups, and these topics do not always attract the same kind of attention that hiring and admissions do. We began by asking participants to share stories of formative mathematical experiences that would not show up on their resumes. Gradually we turned the discussion towards identifying the obstacles that might get in the way of others having those same kinds of experiences. In addition to social pressures, financial obstacles and external commitments emerged as major themes, which naturally led to us thinking about how to lower some of those barriers. Although obviously it is impossible to solve these problems in an hour-long discussion, the informal feedback we received was positive, and it is our hope that participants at least thought for a little bit about some of the subtler factors that can lead to a lack of diversity in mathematics.

3.4 Concluding Presentations

On the last morning of the workshop, August 19th, each group gave a 20-minute presentation highlighting the progress their group had made during the workshop. The atmosphere was collegial and it was a good opportunity for everyone to get a chance to see what the other groups had been doing for the preceding four days. These concluding presentations were also well-attended, despite people having to leave at various times to catch flights.

4 Reports of the Groups

In this section we present brief reports on the scientific progress of each group of researchers. These reports have been written by the leaders of each group and edited by the organizers only for form.

Group 1: Inverse problems and spectral theory of hyperbolic manifolds related to heat kernel rigidity (Report: Gilles Carron)

Group members: Irving Calderón, Gilles Carron, Rafael del Río, Asma Hassannezhad, Sergiu Moroianu, and Xuwen Zhu.

The setting

The questions we study come naturally from the paper [CaTe22]. Here is a link to the paper, and a Beamer exposition of the work may be found here.

The main result of the paper is the following :

Theorem 4.1. *[CaTe22] Let (X, d) be a complete metric space equipped with a non-negative regular Borel measure μ. Assume that there exists a symmetric Dirichlet form $\mathcal{E}$ on (X, μ) admitting a heat kernel p such that for some $\alpha > 0$,*

$$p(x, y, t) = \frac{1}{(4\pi t)^{\alpha/2}} e^{-\frac{\mathsf{d}^2(x,y)}{4t}} \tag{1}$$

holds for any $x, y \in X$ and any $t > 0$. Then α is an integer, (X, d) is isometric to $(\mathbb{R}^\alpha, \mathsf{d}_e)$ where d_e stands for the classical Euclidean distance, and μ is the α-dimensional Hausdorff measure.

An analogous result is also shown for the rigidity of the sphere. The unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ has a natural heat kernel that depends only on the time $t > 0$ and the distance $\mathsf{d}_{\mathbb{S}^n}(x, y)$:

$$e^{-t\Delta_{\mathbb{S}^n}} = K_t^{(n)}(\mathsf{d}_{\mathbb{S}^n}(x, y))$$

for any $x, y \in \mathbb{S}^n$ and $t > 0$. Notice that the Riemannian distance canonically associated with $g_{\mathbb{S}^n}$ satisfies

$$\cos(\mathsf{d}_{\mathbb{S}^n}(x, y)) = \langle x; y \rangle.$$

Theorem 4.2. *[CaTe22] Let (X, d, μ) be a complete metric measure space equipped with a Dirichlet form $\mathcal{E}$ admitting a spherical heat kernel p, that is,*

$$p(x, y, t) = K_t^{(n)}(\mathsf{d}(x, y)) \tag{2}$$

for any $x, y \in X$ and $t > 0$. Then (X, d) is isometric to $(\mathbb{S}^n, \mathsf{d}_{\mathbb{S}^n})$.

The other symmetric spaces of rank 1, namely the compact ones $\mathbb{P}^n(\mathbb{C}), \mathbb{P}^n(\mathbb{H}), \mathbb{P}^2(\mathbb{O})$ and the non-compact ones $\mathbb{H}^n(\mathbb{C}), \mathbb{H}^n(\mathbb{H}), \mathbb{H}^2(\mathbb{O})$, also have heat kernels that depend only on t and on $d(x, y)$. So it is really natural to ask whether the same phenomenon of rigidity takes place for these spaces. However, if there are very useful applications of the rigidity result for the Euclidean case, such as an almost rigidity result, it is not clear that heat kernel rigidity for other non-compact rank 1 symmetric spaces will imply similarly useful results.

The organisation

First on-line meeting: June 16, 2022

We discussed in a document that was sent to all participants on March 31, 2022. After this discussion another document was shared. The document contained some explanations to the questions that have been asked during the first meeting.

Second on-line meeting: July 13, 2022

This meeting was not as useful as the first one, but it helped in order to prepare the workshop.

During the workshop

The discussions alternated between

- Questions for understanding the setting of metric measure spaces endowed with a diffusion operator (Dirichlet space), for which most other participants had almost no intuition coming in,
- Intermediate questions, for instance about the bottom of the spectrum,
- An idea that could lead to a better understanding of Dirichlet space whose heat kernel is (real) hyperbolic. The most promising idea was to look at the Martin boundary and try to relate it to some geometric compactification. This intuition was governed by the fact that the proof of Carron-Tewodrose used the asymptotic cone to capture information at infinity. In this setting we cannot use asymptotic cones (they are not well-defined), but imitating the different compactifications of the hyperbolic space could be useful.

Evaluation

It was not so easy to interact only being online several hours per day, and perhaps the fact the setting was a little too far from the common knowledge of the group was a barrier to moving quickly towards solid directions. Nevertheless, I have the impression that the participants learned a lot of new mathematics. The discussions were very dynamic and I was surprised several times by the very original and interesting proposals formulated during these discussions, by members initially rather distant from the subject.

Group 2: Inverse Steklov Problems on Polygons (Report: Emily Dryden and Carolyn Gordon)

Group members: Emily Dryden, Carolyn Gordon, Javier Moreno, Julie Rowlett, and Carlos Villegas-Blas.

Inverse spectral problems ask the extent to which geometric information about an object is encoded in its spectral data. The Steklov spectrum of a plane domain Ω is the collection of all $\sigma \in \mathbb{R}$ for which there exists $u \in C^\infty(\Omega)$ satisfying

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \partial_\nu u = \sigma u & \text{on } \partial\Omega. \end{cases}$$

where ∂_ν is the normal derivative across the boundary. We denote the eigenvalues by

$$0 = \sigma_0(\Omega) < \sigma_1(\Omega) \leq \sigma_2(\Omega) \leq \dots$$

Striking recent work of Krymski, Levitin, Parnovski, Polterovich and Sher [KLPPS21] addressed the Steklov inverse spectral problem for curvilinear polygons in $\mathbb{R}^2$. Their results on spectral determination, for instance of the number of vertices, the side lengths, and information about the angles, require that the polygons satisfy certain genericity assumptions. When the genericity hypotheses are satisfied, the ordered edge lengths of the curvilinear polygon and the values of $\cos\left(\frac{\pi^2}{\alpha_j^2}\right)$, where $\alpha_1, \dots, \alpha_n$ are the interior angles at the vertices, can be recovered from the Steklov spectrum.

Motivated by their work, our primary goal is to address the following question for various special classes of polygons, e.g., the class of all triangles:

Question. *Are all the polygons in the given class mutually distinguishable by their Steklov spectra; e.g., if two triangles are Steklov isospectral, must they be congruent?*

Thus far we have obtained the following results:

- The Steklov spectrum mutually distinguishes all regular polygons.
- Among all convex polygonal domains, equilateral triangles are Steklov spectrally determined.
- Among all convex quadrilaterals, squares are Steklov spectrally determined.
- The Steklov spectrum mutually distinguishes all isosceles triangles.

As our work has progressed, we've also begun studying eigenvalue bounds for triangles, e.g., we showed:

- For each positive integer k, there exists an explicit constant C_k such that for all triangles T of perimeter one, we have
$$\sigma_k(T) \leq C_k \alpha(T)$$
where $\alpha(T)$ is the smallest interior angle of T.

We hope to obtain an affirmative answer to the following:

Question. *Among all triangles of given perimeter, does the equilateral triangle maximize the first non-zero Steklov eigenvalue σ_1?*

Group 3: Can Dirac boundary conditions (e.g. MIT) be specified using only observable quantities (e.g. currents...)? (Report: Nadine Grosse)

Group members: Xenia Fedosova, Nadine Grosse, Edison Leguizamon, Alejandro Uribe, Hanne van den Bosch, and Angela Vargas.

In this project we examined local self-adjoint boundary conditions for the Dirac operator with a look at their meaning in terms of observables.

The Dirac equation was proposed by Dirac in order to have a covariant relativistic equation for fermions (the Schrdinger equation is not covariant w.r.t. the Lorentz group). The disadvantage is that the spinor itself is not observable but the observables as its probability density or current are bilinear quantities in the spinor.

In the physics literature there are more or less two boundary condition that are used for the Dirac operator. One is the MIT bag condition, the other is mostly used when studying an effective model for graphene is the zig-zag condition. Both are local self-adjoint boundary conditions. But they are not the only possible choice.

In this project we started to examine the very basic question what are local different boundary conditions and whether we can classify them on the level of observables. In the baby case of two-dimensional domains a local self-adjoint boundary condition is in each boundary point a linear combination of MIT and zig-zag. On the level of observables they can be distinguished that the tangential component of the current to the spinor is proportional to the probability density of the spinor with proportionality constant being one for MIT and zero for zig-zig.

In this week we obtained a classification of (sufficiently regular) local self-adjoint boundary condition in general and gave a concrete description in dimension 3 both for the mathematical Dirac operator (acting on $\mathbb{C}^2$-spinors) and the physical Dirac operator (acting on $\mathbb{C}^4$-spinors). We continue to work on this project, in particular on the question from above concerning the observables but also on related questions as the ellipticity of the boundary conditions.

Group 4: Heat asymptotics for Steklov-type problems on manifolds and domains with singular boundaries (Report: Iosif Polterovich)

Group members: Cipriana Anghel, Daniel Grieser, Andrés Felipe Patiño, Camilo Pérez, Iosif Polterovich

This group was composed of three graduate students (Cipriana Anghel from Simion Stoilow Institute of Mathematics, Romania, Andrés Felipe Patiño from Universidad de los Andes, Colombia, and Camilo Pérez from Universidad de los Andes, Colombia) as well as two professors (Daniel Grieser from the University of Oldenburg, Germany, and Iosif Polterovich from Université de Montréal, Canada).

The goal of the group was to investigate heat and eigenvalue asymptotics for Steklov-type problems on manifolds and domains with singular boundaries. Steklov spectral asymptotics for singular geometries have been the subject of several recent papers, see [LPPS22a, LPPS22b, Ivr19, GLPS19, MSS22]. Still, very little is known on this topic in dimensions greater than two.

The group meetings started with student presentations on various aspects of the articles [Ivr19, LPPS22a, MSS22] which helped identify two specific questions the group decided to focus on. The first challenge was to fill in the details in the argument presented in the Appendix in [Ivr19], which is concerned with spectral asymptotics for a certain Dirichlet-to-Neumann type problem for the Helmholtz equation in a planar wedge. This can be viewed as a toy model of the problem that naturally arises after separation of variables in higher dimensions. A certain progress has been made in that direction, however certain points still remain to be clarified. Still, it was observed that results announced in [Ivr19] imply the existence of the heat asymptotics for the Steklov spectrum on domains with edges.

The second question was to extend the formula for the second term in the sloshing eigenvalue asymptotics obtained for triangular prisms in [MSS22] with angles of the form $\pi/2n$, $n \in \mathbb{N}$, to arbitrary angles. Some initial advances have been achieved on this problem, and it was proposed to the student members of the group to continue working on it and to stay in contact with the senior members about their progress.

Group 5: Convergence of Laplacians and singular perturbations (Report: Olaf Post)

Group members: Sabine Bögli, Olivier Bourget, Vladimir Lotoreichik, Olaf Post, Ricardo Weder

As the background of the group was rather broad, we identified four topics the group is working on. We very briefly sketch them in the following:

Dirichlet Laplacians on polygons with non-convex corners

Vladimir Lotoreichik proposed the following problem: consider the Dirichlet Laplacian on a domain in $\mathbb{R}^2$ with one non-convex corner, e.g. an L-shaped domain or a sector of a circle $X = X_\omega = \{\, (r\cos\theta, r\sin\theta) \mid 0 < r < 1, \theta \in (0,\omega) \,\}$ with $\omega \in (\pi, 2\pi)$. It can be shown via the separation of variable ansatz that the Laplacian defined on $\mathsf{H}^2(X) \cap \mathring{\mathsf{H}}^1(X)$ has a 1-parameter family H_α of self-adjoint extensions (different from the Friedrichs extension named H_∞ here).

Question: Can one approximate them by a Dirichlet Laplacian changing the local geometry? Approximation here is meant in the generalised norm resolvent sense developed by J. Weidmann or O. Post. If one keeps the radial symmetry, the problem boils down to the analysis of a family of one-dimensional Schrödinger operators; and probably one has to break the symmetry or even approximate H_α by (pure) Laplacians H_ε on a non-flat family of manifolds $X_{\alpha,\varepsilon}$.

Discrete Laplacians converging to continuous ones

Recently, some progress has been made in norm resolvent convergence result of Schrödinger (or Dirac) operators on $\mathbb{R}^d$ by discretisations on lattices $h\mathbb{Z}^d$, e.g., by Nakamura-Tadano [NaTa21] or Cornean, Garde and Jensen [CGJ21]. The aim is to generalise the results to more general spaces and discretisations. The cited works (and others) make heavy use of the Fourier transform. Spaces we have in mind are for example spaces of the form $X = \mathbb{R}^2 \setminus [0,1]^2$ or $X = (-\infty, 0] \times \mathbb{R} \cup \mathbb{R} \times (-\infty, 0]$ (a three quarter space as in the first project).

Scattering theory and resolvent convergence

We would like to answer the following question: If generalised norm (or strong?) resolvent convergence holds, do the corresponding scattering objects converge? Can one quantify the error (at least in the norm resolvent convergence case)?

The setting is as follows: Assume that $\mathscr{H}_j$ are (separable) Hilbert spaces for $j \in \{1,2\}$. Moreover, assume that H_j are self-adjoint and non-negative operators in $\mathscr{H}_j$ with corresponding unitary groups $U_j(t) := \mathrm{e}^{-\mathrm{i}tH_j}$; and let P_j denote the projection onto the absolutely continuous spectrum of H_j in $\mathscr{H}_j$. Let $R_j := (H_j+1)^{-1}$ be the resolvent. For a bounded operator $J\colon \mathscr{H}_1 \longrightarrow \mathscr{H}_2$ we denote by

$$W_\pm := W_\pm(H_2, H_1, J) := \operatorname{s-lim}_{t\to\pm\infty} U_2(-t) J U_1(t) P_1 \tag{3}$$

the wave operator associated with H_1 and H_2 and identification operator J. Similarly, we have $\widetilde{\mathscr{H}_j}$, $\widetilde{H}_j$, $\widetilde{P}_j$, $\widetilde{R}_j$ $\widetilde{U}_j(t)$, $\widetilde{W}_\pm$ and $\widetilde{J}$.

The idea is to think of the tilded objects as a perturbation of the ones without tilde. One aim is to express the deviation of $\widetilde{W}_\pm$ and $\widetilde{W}_\pm$ in terms of the sandwiched resolvent difference

$$D_j := D(\widetilde{H}_j, H_j, J) := \widetilde{R}_j J_j - J_j R_j$$

for some bounded identification operator $J_j\colon \mathscr{H}_j \longrightarrow \widetilde{\mathscr{H}_j}$. We also need to assume that J_j are close to a unitary operator in the sense that

$$T_j := (\mathrm{id}_{\mathscr{H}_j} - J_j^* J_j) R_j \qquad \text{and} \qquad \widetilde{T}_j := (\mathrm{id}_{\widetilde{\mathscr{H}_j}} - J_j^* J_j^*) \widetilde{R}_j$$

are small.

Question: Can one express e.g. $\|J_2^* \widetilde{W}_\pm J_1 - W_\pm\|$ in terms of $\|T_j\|$ and $\|D_j\|$? Can one use Weidmann's concept of generalised resolvent convergence and embed everything in a common Hilbert space and answer a similar question?

Domain perturbations

Can one define deformations of domains $\Omega_n \to \Omega_\infty$ (e.g. subsets of $\mathbb{R}^d$) such that the corresponding Dirichlet (or Neumann?) Laplacians converge in *norm* resolvent sense (especially *without* compactness assumptions etc.) that ensure for example generalised norm resolvent convergence? In the past mostly conditions ensuring *strong* resolvent convergence have been ensured.

Can we boil it down to properties of Ω_n and $\Omega = \Omega_\infty$?

Group 6: Scattering theory for difference equations with operator coefficients (Report: Luis Silva)

Group members: Abdon Choque, David Sher, Luis Silva, Boris Vertman, and Monika Winklmeier.

Let H be a separable Hilbert space and $\mathcal{H}$ be the space of bi-infinite square-summable sequences with entries in H, namely

$$\mathcal{H} := \ell^2(\mathbb{Z}, H).$$

Consider the sequences $\{A_n\}_{n\in\mathbb{Z}}$ and $\{B_n\}_{n\in\mathbb{Z}}$ of operators in H such that, for every $n \in \mathbb{Z}$, $(I - A_n)$ and B_n are self-adjoint, bounded and compact. Assume also that

$$\sum_{n=-\infty}^{\infty} |n|(\|I - A_n\| + \|B_n\|) < \infty\,. \tag{4}$$

Under these assumptions one defines an operator $\mathcal{T}$ on any sequence $u = \{u_n\}_{n\in\mathbb{Z}}$ in $\mathcal{H}$ by

$$(\mathcal{T}u)_n := A_{n-1}u_{n-1} + B_n u_n + A_n u_{n+1}\,, \quad n \in \mathbb{Z}\,. \tag{5}$$

The aim of this research group is to study the main objects of stationary scattering theory for the operator $\mathcal{T}$ with respect to the operator $\mathcal{T}_0$ defined on $\mathcal{H}$ by

$$(\mathcal{T}_0 u)_n := u_{n-1} + u_{n+1}\,, \quad n \in \mathbb{Z}\,. \tag{6}$$

This problem has been addressed in [CaKa73, Cas73, Cas74, Gus76a, Gus76b, Gus77] when H is one-dimensional (scalar case) and in [AyBa12, BAC16, BFSB21, BFGSB22] when H is finite dimensional. In this project, the main interest is the case of H be infinite dimensional and the operators in the sequences $\{A_n\}_{n\in\mathbb{Z}}$ and $\{B_n\}_{n\in\mathbb{Z}}$ having in general infinite rank. An incipient research on this general setting is given in [Mut20] where some of the complexities of the infinite dimensional case are identified.

In the course of the workshop the research group managed to construct the main objects of the theory, namely, the Jost operator solutions, the transmission and reflection operator coefficients and the scattering matrix. These objects, together with the generalized Wronskians of the Jost solutions, are operator-valued functions (or sequences of them) defined on certain regions of the complex plane and whose properties are crucial for solving the direct and inverse scattering problems pertaining to the operator $\mathcal{T}$.

As a result of the fruitful collaboration during the workshop, the foundations were laid for the development of a research project on various aspects of the scattering theory for the operators mentioned above. The group's participants are currently studying, on the one hand, the analytic properties of the aforesaid operator functions and, on the other hand, the conditions for excluding the existence of accumulation points of the discrete spectrum at the edge of the absolutely continuous spectrum. The results of this work are expected to be reported in a research article.

Group 7: Scattering theory for differential forms and low energy expansions: low regularity domains, relations to cohomology and harmonic analysis (Report: Alexander Strohmaier)

Group members: Clara Aldana, Nelia Charalambous, Dmitry Jakobson, Xenia Spilioti, Alexander Strohmaier, and Amir Vig.

Several of us were participating online only and therefore could not make some of the meetings that ended up late at night in a different time-zone. We did not feel that the time-zone aspect had a severe impact on the group, however.

Aim of the project

On asymptotically Euclidean manifolds and asymptotically conic manifolds low energy resolvent expansions for the Laplace on differential forms are known (see for example [GuSh15], using scattering calculus). More refined statements are available in the special but important case of manifolds Euclidean at infinity with boundary and have been obtained in the long paper [StWa20]. The discrete spectrum of the Laplace operator corresponds to L^2-cohomology ([Car03]) and the expansions of the scattering matrix, the spectral measure, and the generalised eigenfunctions relate to information contained in the L^2-eigenfunctions that represent cohomology classes.

The questions to be looked at are the following:

1. What minimal assumptions on the decay of the metric are needed for such a result to hold; what regularity is needed on both the metric and the boundary?
2. What is the relation between the expansion coefficients and notions such as harmonic capacity from harmonic analysis, and to what extent are related notions for differential forms relevant?
3. For what other manifolds is such a relation expected to hold and what can we conclude from it?
4. To what extent do these expansions carry over to the setting of scattering metrics of the form $g = \frac{dx^2}{x^4} + \frac{h_0(y,dy)}{x^2} + O(x^\infty)$, where x is, as usual, a boundary defining function?

Discussion

In the first meeting we discussed an approximation argument by Tanya Christiansen to approximate the resolvent under metric approximations (as in [Chr99, Proof of Lemma 4.1]). The aim of the discussion was to find out whether the results of [StWa20] can be generalised to the asymptotically Euclidean case by a simple perturbation argument. Whereas we did not find a straightforward way to do that, there was still some sense that under sufficient decay conditions on the metric this should be possible. The discussion then focused on whether or not certain statements about the scattering matrix were true in a more general setting. One example we picked was the result by Melrose and Zworski ([MeZw96]) that the scattering matrix for a scattering manifold M is a Fourier integral operator related to the time π-geodesic flow on the boundary ∂M at infinity. Apparently there is no such statement in the literature for differential forms.

For the Laplace operator on differential p-forms the scattering matrix acts on the space $\Omega^p(\partial M) \oplus \Omega^{p-1}(\partial M)$, rather than functions. The associated Fourier integral operator must therefore be matrix valued and respect the various symmetries present in the problem (Hodge decomposition, Hodge-$*$-operator, exterior differentiation, etc.). We identified a statement that we believe is true for the scattering matrix acting on differential forms and formulated a conjecture that provides a similar statement as for scattering of functions. Some preliminary computations were performed as first steps to prove this conjecture. Work on this is now in progress.

5 Outcome of the Meeting

5.1 Scientific progress and connections

The primary outcome of this meeting consists of the scientific progress made by the ongoing collaborations in the groups. The progress for each group has been discussed in the prior section. Here are some of the highlights:

- At least five of the groups are continuing to work on their open problems. Of these groups, at least four report having made significant new progress during the workshop. Although not all progress will result in publishable work, we expect at least three publications in peer-reviewed journals to result from this workshop.
- Each group included members from Latin America along with participants from outside Latin America. As such, each ongoing collaboration is building a bridge between the Latin American mathematical community and the rest of the world. It is our expectation that these collaborations will in turn lead to further work deepening these mathematical ties.
- Each Ph.D. student participant got the chance to work on an open research problem with a senior mathematician from outside their home institution (and indeed outside their home continent). These students also met a variety of mathematicians from the wider spectral theory community, which should improve their career opportunities in future. At least three of them are also continuing to work in ongoing collaboration in their groups and could gain one or more publications.

5.2 Group-based workshop format

A major question going into this workshop was how well the novel format of research in groups would work. In general, participants appreciated the format and took advantage of the opportunity to broaden their interests. Here is a selection of comments:

- “I really liked this format, as it gave me the opportunity to talk with experts in adjacent areas to mine. This has allowed me to get more familiar with [a new area] much more quickly than if I had tried this on my own. It also gave me ideas for projects in areas that I had not previously considered. ”
- “It is a nice idea to have a workshop where one works on problems rather than presenting proven results.”

- “The workshop was very nicely organised[....] It was a completely new experience for me. I have participated in other collaborative academic events, but in all these events we only studied some modern theories and/or solved some classical exercises about the topic that we were studying. While in this Workshop, it was a real research work in collaboration, in which my team and me could obtain some preliminary results. Moreover, with my Oaxaca workshop team, we are still doing a collaboration even after of the academic event.”

Of particular interest to us was the interaction between Ph.D. students and the more senior mathematicians in their groups. We expected this to be a potential source of tension, as the objectives of introducing graduate students to the field and making quick progress on an open problem are not always aligned. Each group handled this in a different way. Some groups leaned into the former objective. In one group, the students were asked to give presentations on papers they had read in advance of the workshop. The atmosphere in this group, which seems to have been appreciated by the students, was summarized by a senior participant as follows:

“My group was very diverse, with two experienced professors and three graduate students, who among themselves had very different backgrounds. This meant that in order to include everyone in the discussion we had to stick to fairly elementary aspects of the project. This is ok, and we had some interesting discussions, but it was not clear beforehand what to expect. All in all I would describe the group work as more of an intense advisor-graduate student interaction (i.e. an advising situation), rather than making substantial advances on a research problem.”

Other groups chose to jump right in and pull the graduate students along. This sometimes worked well and sometimes did not. Some participants reported that “the rhythm was too fast” for PhD students in their group, and some PhD students felt “unable” to make a “meaningful contribution.” However, there were more positive comments about the diversity of levels than negative ones, with participants saying “it was nice to have groups that included everyone from graduate students to senior researchers”. For example, a senior researcher in one group stated that they “saw the graduate student in our group become more confident about contributing as time went on (especially after the workshop), and that is great!”

Overall the experience for PhD students seems to have been positive, despite a few rough edges. We recommend that organizers of future workshops in this format think carefully about how graduate students fit into the picture and do more to set expectations for the groups in advance. Nevertheless, this was a positive experience for the students, as they were able to meet experts in the field, build professional relationships, work on new projects, and, as one student put it, experience “the way in which [senior colleagues] do research.”

5.3 Hybrid workshop mode

The hybrid mode of this workshop had advantages and disadvantages. On the one hand, it increased safety and also increased accessibility – at least five participants in our informal post-workshop survey indicated that they could not have attended if the workshop had been entirely in person. This is particularly important for giving opportunities to people who do not have significant travel funds and has the potential to help underrepresented folks in mathematics. However, the format also created challenges. Setting meeting times that were convenient to remote participants in various time zones was difficult for some groups. In many cases, the online participants also had to attend to their daily duties at their home institutions and thus could not focus as exclusively on the workshop. A number of participants also noted that the informal social interactions that typically accompany in-person conferences were mostly absent for online participants (although there were attempts, they were not particularly successful). Some participants also mentioned a “loss of spontaneity when discussing ideas” in their groups and in general there is a sense that in-person work is better for communication.

One thing that would help in future in a format such as this would be to set the groups *after* knowing the mode of each participant. In this way groups could either be entirely in person, or either mostly or entirely online. This was not possible under the circumstances of this workshop – a late shift to a hybrid format – but seems promising.

5.4 Conclusion

The workshop “Analytic and Geometric Aspects of Spectral Theory” was successful despite the ongoing challenges posed by COVID-19. It brought together researchers from all over the world who may not have interacted otherwise. Attendees, both in person and online, reported an engaging and rewarding experience. The novel format gave participants a chance to work on problems they might not otherwise have encountered. We think this format offers significant promise and encourage others to pursue it.

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